Special Chapters in the Theory of Analytic Functions of Several Complex Variables

by

B. A. Fuks

Translations of Mathematical Monographs Volume 14

Special Chapters in the Theory of

ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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AMERICAN MATHEMATICAL SOCIETY
PROVIDENCE, RHODE ISLAND
1965

СПЕЦИАЛЬНЫЕ ГЛАВЫ
ТЕОРИИ
АНАЛИТИЧЕСКИХ ФУНКЦИЙ
МНОГИХ КОМПЛЕКСНЫХ
ПЕРЕМЕННЫХ

Б. А. ФУКС

Государственное Издательство
Физико-Математической Литературы

Москва 1963

Translated from the Russian by
A. Jeffrey and N. Mugibayashi

Library of Congress Card Number 65-26324

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PREFACE

The present volume is closely related in its contents to the author's book *Theory of analytic functions of several complex variables*, published in English translation[1] by the American Mathematical Society in 1963. These two volumes together constitute the second edition, considerably revised and enlarged, of the monograph *Theory of analytic functions of several complex variables* published in 1948.

In the second edition, as well as in the first, the author does not aim to cover, even to any extent, all the material which has accumulated in the theory of analytic functions of several complex variables.

The present volume is divided into five chapters: approximation of functions and domains, coherent analytic sheaves and the solution of fundamental problems, domains analytically convex in the sense of Hartogs, holomorphic extension of domains, biholomorphic mappings. In order to understand these chapters, the reader should be familiar with the concepts contained in the first two chapters of *Theory of analytic functions of several complex variables*.

In addition, for §2 of Chapter I the reader must be acquainted with Weil's integral representations (§22, Chapter IV, (I)) and Cousin's first theorem (§25, Chapter V, (I)), for §§8–12 of Chapter II with properties of holomorphically complete complex manifolds (§14.1 and §18. 3–4, Chapter III, (I)), for §14 of Chapter III with Weil's integral representations and the methods of solving Cousin's first problem (throughout §25, Chapter V, (I) and §7, Chapter II) and with properties of sequences of domains (§6, Chapter I), for §§17 and 18 of Chapter IV with the theory of plurisubharmonic functions (§13, Chapter III) and for Chapter V with properties of the kernel function in a domain (§§4 and 5, Chapter I). The remain-

[1] As Volume Eight of the Translations of Mathematical Monographs. In the present text a reference to this book will be indicated by the letter (I) following the number of chapter, section, formula, theorem or cited literature.

ing parts of the present volume, except for certain cross-references, are independent of one another and of Chapters III, IV and V of the first part of the book.

To shorten the volume and simplify the text the proofs of several propositions are not developed in the most general form. For example, Theorems (A) and (B) of H. Cartan are proved for complex manifolds, but not for spaces; the theorem of K. Oka on the domains convex in the sense of Hartogs is proved for the case of spaces of only two complex variables. The "edge of the wedge" theorem of N. N. Bogoljubov is also proved in a simplified form by introducing some hypotheses.

The actual text of the book itself is preceded by an introductory essay giving the most frequently used information from closely related mathematical disciplines. It is recommended that the reader refer to this essay whenever he finds it necessary.

At the request of the author, the first draft of the text of §§8–12, dealing with the theory of coherent analytic sheaves and its application to the solution of the fundamental problem, was written by D. B. Fuks, §19, dealing with the "edge of the wedge" theorem, by V. S. Vladimirov, and §24, dealing with homogeneous bounded domains, by S. G. Gindikin. The latter two sections contain new results which are due to the above-mentioned persons and are introduced here, as a rule, without reference to the original articles.

The author is indebted for advice and a number of valuable remarks to L. A. Aĭzenberg, who looked over the entire text while it was being prepared for the press, and to V. S. Vladimirov, who looked over the text of Chapters III and V.

To all the above persons I wish to express my profound gratitude. I also wish to take this opportunity of thanking other mathematicians who looked over various parts of the book and sent me their suggestions.

Results belonging to many mathematicians are presented in this book. It should be noted, however, that the greatest influence on its contents is due to works of S. Bergman, concerning the kernel function in a domain and its applications, of G. Bremermann, concerning domains convex in the sense of Hartogs and the holomorphic extension of domains, of H. Cartan who established, in collaboration with J.-P. Serre and others, the theory of coherent analytic sheaves and its applications to many important problems of the theory of functions, and of K. Oka, concerning approximation of functions, Cousin's problems and the solution of the *inverse problem of Hartogs.*

June 1962 B. Fuks

FACTS FROM RELATED MATHEMATICAL DISCIPLINES. NOTATION. TERMINOLOGY

The information from related mathematical disciplines given in the present introduction will be necessary for the following exposition and is not to be found in the introductory essay to the first part of the book.

1. Sheaves over a topological space were already considered in the Introduction to the first part of the present book. Here we will introduce a complementary series of concepts. For simplicity, we confine ourselves in our formulation to the case of a sheaf of abelian groups. However, it is easy to see that such definitions may be extended to sheaves consisting of other algebraic objects.

1) Let $\mathfrak{F} = \{\mathfrak{F}_x, \pi, x \in X\}$ be a sheaf of abelian groups over a topological space X. Here $\mathfrak{F}_x$, being an abelian group, is the so-called *sheaf stalk* at a point $x \in X$; $\pi : F \longrightarrow X$ is a *projection* of the space F (the union of elements of the abelian group $\mathfrak{F}_x$, in which the topological structure is defined in a way described below) onto the space X. Then we have $\pi(\mathfrak{F}_x) = x$ and $\pi^{-1}(x) = \mathfrak{F}_x$. We shall often omit the indication on the projection π in the notation of the sheaf.

Take an open subset G of the space F and suppose that every intersection $G \cap \mathfrak{F}_x = \mathfrak{G}_x$ is a subgroup of the group $\mathfrak{F}_x$. Then the sheaf of abelian subgroup $\mathfrak{G} = \{\mathfrak{G}_x, \pi', x \in X\}$ with $\pi' = \pi|_G$ is a *subsheaf* of the sheaf $\mathfrak{F}$. In this case we also say that the sheaf $\mathfrak{G}$ is *imbedded* in the sheaf $\mathfrak{F}$ and write $\mathfrak{G} \subset \mathfrak{F}$. Consider the factor groups $\mathfrak{H}_x = \mathfrak{F}_x/\mathfrak{G}_x$, $x \in X$. In the union of elements of the groups $\mathfrak{H}_x$, we introduce a topology in the following way: a correspondence $h_x : \mathfrak{F}_x \longrightarrow \mathfrak{H}_x$, given at every point $x \in X$, defines a mapping $h : F \longrightarrow H$. A set $U \subset H$ will be regarded as open if and only if its complete inverse image $h^{-1}(U) \subset F$ is an open set in the space F. A sheaf $\mathfrak{H} = \{\mathfrak{H}_x, \pi'', x \in X\}$, with $\pi'' : H \longrightarrow X$ is the projec-

tion of the space H onto the space X and is called the *factor sheaf* of the sheaf $\mathfrak{F}$ by its subsheaf $\mathfrak{G}$. Usually we write $\mathfrak{H} = \mathfrak{F}/\mathfrak{G}$.

2) Let $\mathfrak{F} = \{\mathfrak{F}_x, \pi, x \in X\}$ and $\mathfrak{G} = \{\mathfrak{G}_x, \pi_1, x \in X\}$ be certain sheaves of abelian groups over the space X. If a homomorphism $f_x : \mathfrak{F}_x \longrightarrow \mathfrak{G}_x$ is given at every point $x \in X$ and the mapping $f: F \longrightarrow G$ induced by this homomorphism is continuous, then we say that in this way a *homomorphism* $f: \mathfrak{F} \longrightarrow \mathfrak{G}$ *of the sheaf* $\mathfrak{F}$ *into the sheaf* $\mathfrak{G}$ is defined. This homomorphism is called a *monomorphism* or a *epimorphism* according as the mapping $f_x : \mathfrak{F}_x \longrightarrow \mathfrak{G}_x$ at every point $x \in X$ is monomorphic or epimorphic. We recall that a homomorphism of an abelian group is said to be a monomorphism if it does not map nonzero elements of the group into the zero; it is said to be an epimorphism if it is a surjective mapping. We further note that the monomorphic mapping of the sheaf $\mathfrak{F}$ into the sheaf $\mathfrak{G}$ is an *imbedding* of the sheaf $\mathfrak{F}$ into the sheaf $\mathfrak{G}$. A homomorphism which is simultaneously monomorphic and epimorphic is an *isomorphism.*

For example, if the sheaf $\mathfrak{G}$ is a subsheaf of the sheaf $\mathfrak{F}$, then the imbeddings $i_x : \mathfrak{G}_x \longrightarrow \mathfrak{F}_x$ of the subsheaf $\mathfrak{G}_x$ into the sheaf $\mathfrak{F}_x$ at all points $x \in X$ define, in their totality, a monomorphism $i: \mathfrak{G} \longrightarrow \mathfrak{F}$. Conversely, any monomorphism is shown to be an imbedding of the subsheaf into the sheaf.

Another example: the collection of the mappings $h_x : \mathfrak{F}_x \longrightarrow \mathfrak{H}_x$ at all points $x \in X$ (used for the construction of the factor sheaf) defines an epimorphism $h: \mathfrak{F} \longrightarrow \mathfrak{H}$, which is usually called the projection or the *factorization* of the sheaf $\mathfrak{F}$. Conversely, any epimorphism may be represented as a factorization.

3) Consider a homomorphism $f: \mathfrak{F} \longrightarrow \mathfrak{G}$. The subsheaf $\mathfrak{F}^0 = \{\mathfrak{F}^0_x, \pi', x \in X\}$ of the sheaf $\mathfrak{F}$ is called the *kernel* $\operatorname{Ker} f$ of that homomorphism. The space F^0 of this subsheaf consists of only those elements of the space F which are carried into the zero of the group $\mathfrak{G}_x$ under the homomorphism f. The subsheaf $f(\mathfrak{F})$ of the sheaf $\mathfrak{G}$ is called the *image* $\operatorname{Im} f$ of the homomorphism $f: \mathfrak{F} \longrightarrow \mathfrak{G}$ and the image $f(F) \subset G$ under the mapping f serves as its space. The factor sheaf $\mathfrak{G}/f(\mathfrak{F})$ is called the *cokernel* $\operatorname{Coker} f$ of the homomorphism $f: \mathfrak{F} \longrightarrow \mathfrak{G}$. In this way, kernel, image and cokernel of the homomorphism of the sheaf are defined in the same way as the corresponding concepts for groups.

4) Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be two subsheaves of a sheaf $\mathfrak{F}$, and let G_1 and G_2 be spaces of these subsheaves. A subsheaf $\mathfrak{G}$ of the sheaf $\mathfrak{F}$ with the space $G = G_1 \cap G_2$ is called the *intersection* of the subsheaves $\mathfrak{G}_1$ and $\mathfrak{G}_2$, and is denoted by the symbol $\mathfrak{G}_1 \cap \mathfrak{G}_2$.

5) We recall that a finite or infinite set of groups $\{\mathfrak{F}^\nu,\ \nu = 0, 1, 2, \cdots\}$ and homomorphisms $f^\nu : \mathfrak{F}^\nu \longrightarrow \mathfrak{F}^{\nu+1}$ is called a sequence of groups. This sequence is usually written as

$$\cdots \xrightarrow{f^{\nu-1}} \mathfrak{F}^\nu \xrightarrow{f^\nu} \mathfrak{F}^{\nu+1} \xrightarrow{f^{\nu+1}} \cdots. \tag{0.1}$$

The sequence (0.1) is said to be *exact*, provided an element $\alpha \in \mathfrak{F}^\nu$ is mapped under the homomorphism f^ν into the zero element of the group $\mathfrak{F}^{\nu+1}$, if and only if it is the image of a certain element $\beta \in \mathfrak{F}^{\nu-1}$ under the homomorphism $f^{\nu-1}$, i.e., if $\operatorname{Im} f^{\nu-1} = \operatorname{Ker} f^\nu$.

These definitions may be extended to the case of sheaves of abelian groups. In this connection it is natural to impose the additional requirement that the collection of the homomorphisms $f^\nu_x : \mathfrak{F}^\nu_x \longrightarrow \mathfrak{F}^{\nu+1}_x$, $x \in X$, should define a homomorphism $f^\nu : \mathfrak{F}^\nu \longrightarrow \mathfrak{F}^{\nu+1}$ of corresponding sheaves. The fulfillment of the relation $\operatorname{Im} f^{\nu-1} = \operatorname{Ker} f^\nu$ for all values of the index ν is a condition for the *exactness of the sequence of sheaves* $\{\mathfrak{F}^\nu\}$.

6) Let $\mathfrak{F}^1 = \{\mathfrak{F}^1_x, \pi_1, x \in X\}$, $\mathfrak{F}^2 = \{\mathfrak{F}^2_x, \pi_2, x \in X\}$ be two sheaves over the space X, and let F^1 and F^2 be spaces of these sheaves. From the product $F^1 \times F^2$ of these spaces we single out a subset F consisting of those elements (f^1, f^2), with $f^1 \in F^1$, and $f^2 \in F^2$, for which $\pi_1(f^1) = \pi_2(f^2)$. In the set F we introduce the topology induced from the space $F^1 \times F^2$ and, in what follows, we will speak of the space F. The projection $\pi : F \longrightarrow X$ is defined in a natural way by means of the projections π_1 and π_2. The group $\pi^{-1}(x) = \pi_1^{-1}(x) \times \pi_2^{-1}(x)$ turns out to be the inverse image of each point $x \in X$ under the projection π. The sheaf $\mathfrak{F} = \{\pi^{-1}(x), \pi, x \in X\}$ is called the *direct sum of the sheaves* $\mathfrak{F}^1$ and $\mathfrak{F}^2$. In this case we write $\mathfrak{F} = \mathfrak{F}^1 + \mathfrak{F}^2$. We observe that each of the sheaves $\mathfrak{F}^1$ and $\mathfrak{F}^2$ is a subsheaf of the sheaf $\mathfrak{F}$, and that the factor sheaf $\mathfrak{F}/\mathfrak{F}^1$ and the sheaf $\mathfrak{F}^2$ on one hand, and the factor sheaf $\mathfrak{F}/\mathfrak{F}^2$ and the sheaf $\mathfrak{F}^1$ on the other, consist of isomorphic groups (at every point $x \in X$).

7) In the sequel we shall also use some other facts from the theory of sheaves. These can be found, for example, in the book of R. Godement, *Théorie des faisceaux,* Actualités Sci. Ind. No. 1252, Hermann, Paris, 1958, in the book of F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, Berlin, 1956, or in the article of J.-P. Serre, *Faisceaux algébriques cohérents,* Ann. of Math. (2) 61 (1955), 197–278.

2. Cohomology with coefficients in a group. Let X be a paracompact Hausdorff space with a countable basis of open sets, let $\mathfrak{U} = \{U_i, i \in I\}$ be an at most count-

able open covering of that space, and let G be an abelian group. Take some finite collection of indices $s = (i_0, \ldots, i_k)$ from the set I and put $U_s = U_{i_0 \ldots i_k} = \bigcap_{j=0}^{k} U_{i_j}$. We suppose that a correspondence

$$\phi : U_s \to \phi_s \in G$$

is given in a certain way. Here $\phi_s = \epsilon\phi_{\tau s}$ for an arbitrary permutation τ of the set of indices s with $\epsilon = \pm 1$ for even and odd permutations, respectively (in particular, $\phi_s = 0$ if any two indices belonging to s are identical). Such a correspondence is called a k-dimensional (skew-symmetric) *cochain* ϕ, corresponding to the covering $\mathfrak{U}$ of the space X, and the element ϕ_s is the value of that cochain on the intersection U_s.

Let $s = (i_0, \ldots, i_{k+1})$, $s_j = (i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{k+1})$. We form a $(k+1)$-dimensional cochain $\delta\phi$ taking on the value

$$(\delta\phi)_s = \sum_{j=0}^{k+1} (-1)^{j+1} \phi_{s_j}$$

on the intersection $U_{i_0 \cdots i_{k+1}}$.

The cochain $\delta\phi$ is called the *coboundary* of the cochain ϕ. A cochain ϕ, for which $\delta\phi = 0$, is called a *cocycle*; a k-dimensional cochain $\phi = \delta\gamma$, where γ is a certain $(k-1)$-dimensional cochain, is called a *coboundary*. It is easy to prove that for any cochain ϕ the cochain $\delta\delta\phi = 0$. Now suppose that there are given two k-dimensional cochains. By considering those elements of the group G which are the sum of the values of these cochains on one and the same intersection, we define a new cochain. It is called the sum of initial cochains. Under this addition the k-dimensional cochains form an abelian group which will be denoted by the symbol $C^k(\mathfrak{U}, G)$.

A homomorphism $\delta : C^k(\mathfrak{U}, G) \to C^{k+1}(\mathfrak{U}, G)$ is called a *coboundary operator*. We denote the kernel of this homomorphism by $Z^k(\mathfrak{U}, G)$, and its image by $B^{k+1}(\mathfrak{U}, G)$. The relation $\delta\delta\phi = 0$ is equivalent to $B^k(\mathfrak{U}, G) \subset Z^k(\mathfrak{U}, G)$. It is evident that k-dimensional cocycles are elements of the kernel $Z^k(\mathfrak{U}, G)$ and k-dimensional coboundaries are elements of the image $B^k(\mathfrak{U}, G)$.

The factor group $Z^k(\mathfrak{U}, G)/B^k(\mathfrak{U}, G)$ is called the *k-dimensional cohomology group of the covering* $\mathfrak{U}$ *of the space* X *with coefficients in the group* G, and is denoted by the symbol $H^k(\mathfrak{U}, G)$. The *cohomology classes* are used for elements of the group $H^k(\mathfrak{U}, G)$. Two cocycles ϕ and γ belonging to the same cohomology class are said to be cohomologous to each other; in this case we write $\phi \sim \gamma$.

If all k-dimensional cocycles are k-dimensional coboundaries (then the group $Z^k(\mathfrak{U}, G)$ consists of a single cohomology class), the group $H^k(\mathfrak{U}, G)$ consists of a single (zero) element. Then we write $H^k(\mathfrak{U}, G) = 0$ and call this group *trivial.*

Let $\mathfrak{V} = \{V_j, j \in J\}$ be another open covering of the space X. The covering $\mathfrak{V}$ is said to be a *refinement* of the covering $\mathfrak{U}$, if there exists a correspondence of the indices $\tau : J \longrightarrow I$ such that $V_j \subset U_{\tau(j)}$ for every $j \in J$. In this case we write $\mathfrak{V} \prec \mathfrak{U}$ and say that a relation of *partial ordering* is given between the coverings $\mathfrak{V}$ and $\mathfrak{U}$.

The correspondence τ defines a homomorphism $\tilde{\tau} : C^k(\mathfrak{U}, G) \longrightarrow C^k(\mathfrak{V}, G)$ permutable with the homomorphism δ. It means that the homomorphism $\tilde{\tau}$ carries cocycles into cocycles and cohomology cocycles into cohomology cocycles. Therefore the homomorphism $\tilde{\tau}$ in its turn induces a homomorphism $\tau^* : H^k(\mathfrak{U}, G) \longrightarrow H^k(\mathfrak{V}, G)$. Furthermore, it can be shown that the homomorphism τ^* is completely defined by a pair of coverings $\mathfrak{U}$ and $\mathfrak{V}$, where $\mathfrak{V} \prec \mathfrak{U}$. Accordingly, in the following we will write $\tau^* = \sigma(\mathfrak{V}, \mathfrak{U})$. Finally we observe that $\tau^* = \sigma(\mathfrak{V}, \mathfrak{U})$ has the following properties: $\sigma(\mathfrak{U}, \mathfrak{U})$ is an identity automorphism; if $\mathfrak{W} \prec \mathfrak{V} \prec \mathfrak{U}$, then $\sigma(\mathfrak{W}, \mathfrak{V}) \circ \sigma(\mathfrak{V}, \mathfrak{U}) = \sigma(\mathfrak{W}, \mathfrak{U})$.

These circumstances allow us to get an inductive (direct) limit of the groups $H^k(\mathfrak{U}, G)$ with respect to a directed set of countable coverings of the space X. The limit thus obtained is denoted by the symbol $H^k(X, G)$ and is called the *k-dimensional cohomology group of the space X with coefficients in the group G.*

We recall that a partially ordered set A is said to be *directed* if, for arbitrary elements $\alpha, \beta \in A$, one can find an element $\gamma \in A$ such that $\gamma \prec \alpha, \gamma \prec \beta$. Suppose that we have a system of abelian groups $\{G_\alpha, \alpha \in A\}$ and homomorphisms $f_\alpha^\beta : G_\alpha \longrightarrow G_\beta$, given for $\alpha, \beta \in A$, $\beta \prec \alpha$ and satisfying the condition $f_\alpha^\beta f_\beta^\gamma = f_\alpha^\gamma$ for $\gamma \prec \beta \prec \alpha$. Such a system is called a *direct spectrum of the group.*

In the union $\bigcup_{\alpha \in A} G_\alpha$ we introduce an equivalence relation. Elements $g_\alpha \in G_\alpha$ and $h_\beta \in G_\beta$ will be regarded as equivalent if and only if there exists an element $\gamma \in A$, satisfying the condition $\gamma \prec \alpha$, $\gamma \prec \beta$, such that $f_\alpha^\gamma g_\alpha = f_\beta^\gamma h_\beta$. This equivalence relation is reflexive, symmetric and transitive. The set of equivalence classes in the union $\bigcup_{\alpha \in A} G_\alpha$ will be denoted by G. If $g, h \in G$, and $g_\alpha \in G_\alpha$, $h_\beta \in G_\beta$ are representatives of these classes, then $g + h$ is an equivalence class, to which there belongs the element $f_\alpha^\gamma g_\alpha + f_\beta^\gamma h_\beta \in G_\gamma$ (here $\gamma \prec \alpha$, $\gamma \prec \beta$).

Under the above operation of addition, the set G becomes an abelian group, which is called the *inductive (direct) limit of the groups $\{G_\alpha\}$ with respect to the directed set A.* In this case we write $G = \lim \operatorname{ind} G_\alpha$.

In subsection 8 of the Introduction to the first part of the present book, we defined cohomologies of a manifold X of the class $\mathcal{C}^1$ by means of exterior differential forms. We have

THEOREM 0.1 (de Rham). *The cohomologies of a manifold X of the class $\mathcal{C}^1$, calculated by means of complex-valued exterior differential forms of degree k, coincide with the groups $H^k(X, C)$, where C is the additive group of complex numbers.*

3. Cohomologies with coefficients in a sheaf. Let X be a paracompact space with a countable basis of open sets and let $\mathfrak{F}(X)$ be a sheaf of abelian groups given over that space. Preserving the notations of the preceding subsection, we consider a correspondence

$$f: U_s \longrightarrow f_s \in \mathfrak{F}_{U_s},$$

where $\mathfrak{F}_{U_s}$ is, as always, the group of the section of the sheaf $\mathfrak{F}(X)$ over U_s, and $f_s = \epsilon f_{\tau s}$. Such a correspondence is called a k-dimensional *cochain* f of a covering $\mathfrak{U}$. As in the preceding subsection, we define the addition operation of k-dimensional cochains and form the group $C^k(\mathfrak{U}, \mathfrak{F})$ of k-dimensional cochains. We construct a $(k+1)$-dimensional cochain δf, assuming on the intersection $U_{i_0 \ldots i_{k+1}}$ the value

$$(\delta f)_s = \sum_{j=0}^{k+1} (-1)^{j+1} \rho_j f_{s_j},$$

where again $s = (i_0, \ldots, i_{k+1})$, $s_j = (i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{k+1})$. Here $\rho_j : \mathfrak{F}_{U_{s_j}} \longrightarrow \mathfrak{F}_{U_s}$ are natural projections, putting into correspondence with each section of the sheaf $\mathfrak{F}(X)$ over U_{s_j} its boundary over U_s. The cochain δf is called the *coboundary* of the cochain f, the homomorphism $\delta : C^k(\mathfrak{U}, \mathfrak{F}) \longrightarrow C^{k+1}(\mathfrak{U}, \mathfrak{F})$ is called the *coboundary* operator, the cochain f, for which $\delta f = 0$, is called the *cocycle*; the k-dimensional cochain $f = \delta g$, where $g \in C^{k-1}(\mathfrak{U}, \mathfrak{F})$, is called the *coboundary*. For an arbitrary cochain f, the cochain $\delta\delta f = 0$.

Cocycles and coboundaries make up subgroups $Z^k(\mathfrak{U}, \mathfrak{F})$ and $B^k(\mathfrak{U}, \mathfrak{F})$ of the group $C^k(\mathfrak{U}, \mathfrak{F})$. The relation $\delta\delta f = 0$ is equivalent to the inclusion relation $B^k(\mathfrak{U}, \mathfrak{F}) \subset Z^k(\mathfrak{U}, \mathfrak{F})$. The factor group $H^k(\mathfrak{U}, \mathfrak{F}) = Z^k(\mathfrak{U}, \mathfrak{F}) / B^k(\mathfrak{U}, \mathfrak{F})$ is called the *cohomology group of the covering with coefficients in the sheaf* $\mathfrak{F}(X)$; cohomology classes are its elements.

We now consider countable coverings $\mathfrak{U} = \{U_i, i \in I\}$ and $\mathfrak{V} = \{V_j, j \in J\}$ of a space X and assume that a correspondence of indices $\tau : J \longrightarrow I$ establishes the relation $\mathfrak{V} < \mathfrak{U}$. Let $f \in C^k(\mathfrak{U}, \mathfrak{F})$ be a k-dimensional cochain of the covering $\mathfrak{U}$.

By $\tilde{\tau} f \in C^k(\mathfrak{V}, \mathfrak{F})$ we denote a k-dimensional cochain of the covering $\mathfrak{V}$, for which

$$(\tilde{\tau} f)_{j_0 \cdots j_k} = \rho(f_{\tau(j_0) \cdots \tau(j_k)}),$$

where $\rho: \mathfrak{F}_{U_{\tau(j_0)} \cdots \tau(j_k)} \longrightarrow \mathfrak{F}_{V_{j_0} \cdots j_k}$ is a natural homomorphism.

As in the preceding subsection, the correspondence $f \longrightarrow \tilde{\tau} f$ defines a homomorphism $\tilde{\tau}: C^k(\mathfrak{U}, \mathfrak{F}) \longrightarrow C^k(\mathfrak{V}, \mathfrak{F})$, permutable with the homomorphism δ. Therefore the homomorphism $\tilde{\tau}$ in its turn defines a homomorphism $\tau^*: H^k(\mathfrak{U}, \mathfrak{F}) \longrightarrow H^k(\mathfrak{V}, \mathfrak{F})$. This homomorphism τ^* is completely defined by a pair of coverings $\mathfrak{U}, \mathfrak{V}$, where $\mathfrak{V} < \mathfrak{U}$. This last circumstance makes it possible to obtain $\lim \operatorname{ind} H^k(\mathfrak{U}, \mathfrak{F})$ with respect to a directed set of countable open coverings of the space X; this limit is denoted by the symbol $H^k(X, \mathfrak{F})$ and is called the *k-dimensional cohomology group of the space X with coefficients in the sheaf* $\mathfrak{F}(X)$.

We note that from the method of defining the group $H^k(X, \mathfrak{F})$ follows the existence of a homomorphism $\phi_{\mathfrak{U}}: H^k(\mathfrak{U}, \mathfrak{F}) \longrightarrow H^k(X, \mathfrak{F})$ with the following properties: if ξ is an element of $H^k(\mathfrak{U}, \mathfrak{F})$ and $\phi_{\mathfrak{U}}(\xi) = 0$, then one can find a covering $\mathfrak{V} < \mathfrak{U}$ such that $\phi_{\mathfrak{V}} = \sigma(\mathfrak{V}, \mathfrak{U}) \phi_{\mathfrak{U}}$, $\sigma(\mathfrak{V}, \mathfrak{U})(\xi) = 0$. In this connection, for each element $\eta \in H^k(X, \mathfrak{F})$, there exists a covering $\mathfrak{U}$ and an element $\xi \in H^k(\mathfrak{U}, \mathfrak{F})$ such that $\phi_{\mathfrak{U}}(\xi) = \eta$.

4. Properties of cohomology with coefficients in a sheaf.

THEOREM 0.2. $H^0(X, \mathfrak{F}) = \mathfrak{F}_X$, *i.e., a null-dimensional cohomology group of a space X with coefficients in a sheaf* $\mathfrak{F}(X)$ *coincides with a group of the sections of* $\mathfrak{F}(X)$ *over the space X.*

In fact, the null-dimensional cochain f of the covering $\mathfrak{U}$ puts into correspondence with each element of the covering $\mathfrak{U}$ the section of the sheaf $\mathfrak{F}(X)$ over this element. The condition $\delta f = 0$ means that in the intersection of two elements these sections over them coincide. Because of the circumstance mentioned above, groups $Z^0(\mathfrak{U}, \mathfrak{F})$ and $\mathfrak{F}_X$ are isomorphic. Our assertion now follows from the facts that $H^0(\mathfrak{U}, \mathfrak{F}) = Z^0(\mathfrak{U}, \mathfrak{F})$, and for $\mathfrak{V} < \mathfrak{U}$

$$\phi_{\mathfrak{V}} = \phi_{\mathfrak{U}} \sigma(\mathfrak{V}, \mathfrak{U})$$

(i.e., the homomorphism $\sigma(\mathfrak{V}, \mathfrak{U})$ is an isomorphism).

We also note that $H^k(X, \mathfrak{F}) = 0$ if $k > \operatorname{Dim} X$.

We consider a homomorphism of sheaves $\phi: \mathfrak{F}(X) \longrightarrow \mathfrak{G}(X)$ and a cochain $f \in C^k(\mathfrak{U}, \mathfrak{F})$. The formula $(\phi f)_s = \phi(f_s)$ defines a cochain $\phi f \in C^k(\mathfrak{U}, \mathfrak{G})$. Evidently the mapping $f \longrightarrow \phi f$ defines a homomorphism $\phi: C^k(\mathfrak{U}, \mathfrak{F}) \longrightarrow C^k(\mathfrak{U}, \mathfrak{G})$.

This last homomorphism is permutable with the coboundary operator δ and hence induces a homomorphism $\tilde{\phi}: H^k(\mathfrak{U}, \mathfrak{F}) \longrightarrow H^k(\mathfrak{U}, \mathfrak{G})$. It is easy to see that if $\mathfrak{U}$ and $\mathfrak{V}$ are coverings and $\mathfrak{V} \prec \mathfrak{U}$, then $\phi^* \circ \sigma(\mathfrak{V}, \mathfrak{U}) = \sigma(\mathfrak{V}, \mathfrak{U}) \circ \phi^*$. Accordingly, going over to the inductive limit, we obtain a homomorphism

$$\phi^*: H^k(X, \mathfrak{F}) \longrightarrow H^k(X, \mathfrak{G}).$$

If there are given homomorphisms $\phi_k: \mathfrak{F}(X) \longrightarrow \mathfrak{G}_k(X),' \; k = 1, 2,$ then it is natural to define a homomorphism $\phi_1 + \phi_2: \mathfrak{F}(X) \longrightarrow \mathfrak{G}_1(X) + \mathfrak{G}_2(X)$. It is easily seen that $(\phi_1 + \phi_2)^* = \phi_1^* + \phi_2^*$.

If there are given homomorphisms $\phi: \mathfrak{F}(X) \longrightarrow \mathfrak{G}(X)$, and $\psi: \mathfrak{G}(X) \longrightarrow \mathfrak{H}(X)$, then $(\psi \circ \phi)^* = \psi^* \circ \phi^*$.

Finally we note that $1^* = 1$. Here 1 is the identity homomorphism. Suppose that

$$0 \longrightarrow \mathfrak{A}(X) \xrightarrow{\alpha} \mathfrak{B}(X) \xrightarrow{\beta} \mathfrak{C}(X) \longrightarrow 0 \tag{0.2}$$

is an exact sequence of sheaves over the space X. If V is a neighborhood of a a point $x \in X$ and $f \in \mathfrak{C}_V$ is the section of the sheaf $\mathfrak{C}(X)$ over V, then there exists a neighborhood $U \subset V$ of that point x and the section $g \in \mathfrak{B}_U$ of the sheaf $\mathfrak{B}(X)$ over U such that $\beta g = f$ over the neighborhood U. Let $\mathfrak{V} = \{V_i, \; i \in I\}$ be a locally finite covering of the space X, let $\phi \in H^k(\mathfrak{V}, \mathfrak{C})$ be a cohomology class and let f be a representative of its cocycle. Since the covering $\mathfrak{V}$ is locally finite, there exists a covering $\mathfrak{W} = \{W_i, \; i \in I\}$ such that $\overline{W}_i \subset V_i$. For each point $x \in X$ we choose a neighborhood U_x so that for all indices $i \in I$:

1) If $x \in V_i$ (accordingly $x \in W_i$), then $U_x \subset V_i$ (accordingly $U_x \subset W_i$).

2) If $U_x \cap W_i \neq \emptyset$, then $U_x \subset V_i$.

3) If $x \in V_{i_0 \ldots i_k}$, then there exists an element $g \in \mathfrak{B}_{U_x}$ such that $\beta(g) = f_{i_0 \ldots i_k}$ over the neighborhood U_x.

The neighborhoods $\{U_x, \; x \in X\}$ form a certain covering of the space X.

By the use of the fact that the space X has a countable basis of open sets, we single out from this covering an at most countable covering $\mathfrak{U} = \{U_j, \; j \in J\}$ of the space X. Evidently $\mathfrak{U} \prec \mathfrak{V}$; let $\tau: J \longrightarrow I$ be a correspondence of the indices, for which $U_j \subset V_{\tau(j)}$. In view of the conditions 1)–3) there exists a cochain $g \in C^k(\mathfrak{U}, \mathfrak{B})$ such that $\beta(g) = \tau f$. Since $\delta f = 0$, we have $\delta\beta(g) = \beta(\delta g) = 0$. Therefore, for an arbitrary finite system of indices $s = (j_0, \ldots, j_k)$,

$$[\beta(\delta g)]_s = \beta(\delta g)_s = 0.$$

From the exactness of the sequence (0.2) it follows that $(\delta g)_s = \alpha(h_s)$, where $h_s \in \mathfrak{A}_{U_s}$ is the section of the sheaf $\mathfrak{A}(X)$ over U_s. By the same argument a cochain $h \in C^{k+1}(\mathfrak{U}, \mathfrak{A})$ is also defined. By the method of construction, $\delta(\alpha(h)) = 0$, i.e., $\alpha(\delta h) = 0$. From the last equation it further results that $(\delta h)_s = 0$, i.e. $\delta h = 0$. Accordingly the cochain h is a cocycle. This cocycle represents a cohomology class $\mathfrak{g}\phi \in H^{k+1}(X, \mathfrak{C})$. It is easy to see that this last class is completely defined by the cohomology class $\phi \in H^k(X, \mathfrak{A})$ (hence we have denoted it by the symbol $\mathfrak{g}\phi$). The homomorphism thus defined

$$\mathfrak{g}: H^k(X, \mathfrak{C}) \longrightarrow H^{k+1}(X, \mathfrak{A})$$

is called *Bockstein's homomorphism*. We have

THEOREM 0.3. *The sequence*

$$\ldots \to H^k(X, \mathfrak{A}) \xrightarrow{\alpha^*} H^k(X, \mathfrak{B}) \xrightarrow{\beta^*} H^k(X, \mathfrak{C}) \xrightarrow{\mathfrak{g}} H^{k+1}(X, \mathfrak{A}) \to \ldots \tag{0.3}$$

is exact.

We note that the assertion of Theorem 0.3 in essence unifies the following six statements:

1) Ker $\beta^* \subset$ Im α^*, i.e., $(\beta^* \circ \alpha^*)\phi = 0$. Here $\phi \in H^k(X, \mathfrak{A})$, $0 \in H^k(X, \mathfrak{C})$.

2) Ker $\beta^* \supset$ Im α^*, i.e., if $\phi \in H^k(X, \mathfrak{B})$ and $\beta^*\phi = 0$, then there exists an element $\psi \in H^k(X, \mathfrak{A})$ such that $\alpha^*\psi = \phi$.

3) Ker $\mathfrak{g} \subset$ Im β^*, i.e., $(\mathfrak{g} \circ \beta^*)\phi = 0$. Here $\phi \in H^k(X, \mathfrak{B})$, $0 \in H^{k+1}(X, \mathfrak{A})$.

4) Ker $\mathfrak{g} \supset$ Im β^*, i.e., if $\phi \in H^k(X, \mathfrak{C})$ and $\mathfrak{g}\phi = 0$, then there exists an element $\psi \in H^k(X, \mathfrak{B})$ such that $\beta^*\psi = \phi$.

5) Ker $\alpha^* =$ Im $\mathfrak{g}$, i.e., $(\alpha^* \circ \mathfrak{g})\phi = 0$. Here $\phi \in H^k(X, \mathfrak{C})$, $0 \in H^{k+1}(X, \mathfrak{B})$.

6) Ker $\alpha^* \supset$ Im $\mathfrak{g}$, i.e., if $\phi \in H^{k+1}(X, \mathfrak{A})$ and $\alpha^*\phi = 0$, then there exists an element $\psi \in H^k(X, \mathfrak{C})$, such that $\mathfrak{g}\psi = \phi$.

We shall refer below to the following proposition:

THEOREM 0.4. *If F is an abelian group and $\mathfrak{F}(X)$ is a fixed sheaf of groups isomorphic to the group F over a paracompact space X, then $H^k(X, F) = H^k(X, \mathfrak{F})$ for all $k \geq 0$.*

Further information on the theory of cohomologies with coefficients in a sheaf can be found in the literature mentioned at the end of subsection 1 of the present Introduction.

5. Homology group and Betti numbers. In subsection 6 of the Introduction to

the first part of the present book, we defined a group of homologies with complex coefficients in a manifold X, whose dimension is equal to that of the manifold. We shall need in the sequel a homology group $H_k(X_m, \mathfrak{Z})$ of an arbitrary dimension k for an oriented m-dimensional smooth manifold X_m, where $\mathfrak{Z}$ is the group of (real) integers. These homology groups are defined analogously to cohomology groups. The difference consists in that 1) instead of the group of cochains one considers the group of finite k-dimensional chains, which are defined by means of k-dimensional simplexes in the same way as in the Introduction to the first part of the present book, where the chains of dimension equal to that of the manifold were defined; 2) instead of the coboundary operator δ, raising the dimension of the cochain by unity, one takes a boundary operator ∂, lowering the dimension of the chain by unity.

We consider the case when the group $H_k(X_m, \mathfrak{Z})$ has a finite set of generators. The maximum number q of linearly independent elements $\alpha_1, \cdots, \alpha_q \in H_k(X_m, \mathfrak{Z})$ (i.e., elements such that for arbitrary integers $\nu_1, \cdots, \nu_q$, not all being equal to zero, the cycle $\nu_1\alpha_1 + \cdots + \nu_q\alpha_q$ is not homologous to zero), in other words the maximum number of the homologically independent k-dimensional cycles, is called the *k-dimensional Betti number* of the manifold X_m.

Let γ be a k-dimensional cycle on the manifold X_m, ϕ a closed differential form of degree k. Then $\int_\gamma \phi = 0$ if the cycle $\gamma \sim 0$, i.e., if it is a boundary of a certain chain (this follows from Stokes' theorem, see, for example, §20.2, Chapter II, (I)).

The inverse statement is false. If for some natural number $\nu \neq 0$ the cycle $\nu\gamma \sim 0$, then it does not follow from this that the cycle $\gamma \sim 0$, but that $\int_\gamma \phi = \int_{\nu\gamma} (1/\nu)\phi = 0$. One can prove (see, for example, G. de Rham, *Variétés différentiables*, Actualités Sci. Ind. No. 1222, Hermann, Paris, 1960) that, in general, if for some cycle γ the integral $\int_\gamma \phi = 0$, ϕ being an arbitrary closed form, then there exists a natural number $\nu \neq 0$ such that $\nu\gamma \sim 0$.

If for the manifold X_m the cohomology group $H^k(X_m, C) = 0$, then in view of Theorem 0.1 every closed form ϕ of degree k, defined on that manifold, is a differential and by Stokes' theorem $\int_\gamma \phi = 0$ for an arbitrary k-dimensional cycle γ. Then there exists a natural number ν such that $\nu\gamma \sim 0$. Thus we have proved

THEOREM 0.5. *If for smooth manifolds X_m the cohomology group $H^k(X_m, C) = 0$, then in a homology group $H_k(X_m, \mathfrak{Z})$ all elements are of finite order. If a set of generators of the group $H_k(X_m, \mathfrak{Z})$ is finite, then the k-dimensional*

Betti number of the manifold X_m *is equal to zero.*

6. Generalized functions and their Fourier transformation. In the following exposition we shall use the notion of a generalized function (or a distribution of L. Schwartz). By a generalized function we understand a linear continuous functional over a space S of the basic functions of Sobolev-Schwartz. Here the space S consists of complex-valued functions $\phi(\xi) \in \mathcal{C}^\infty$, defined in the space $R_n^{(\xi)}$ of real variables $\xi_1, \dots, \xi_n$. As to these functions we assume that as a point ξ tends to infinity they decrease, along with all their derivatives, faster than any power of $|\xi|^{-1}$, where $|\xi| = (\xi_1^2 + \dots + \xi_n^2)^{1/2}$. A topology in the space S is introduced by means of a countable set of norms

$$\|\varphi\|_p = \sup_{\|\alpha\| \leqslant p,\, \xi \in R_n} (1+|\xi|)^p |\partial^\alpha \varphi(\xi)|, \qquad p = 0,\ 1,\ 2, \dots$$

Here $\|\alpha\| = \alpha_1 + \dots + \alpha_n$; $\partial^\alpha \phi(\xi) = \partial^{\|\alpha\|}\phi / \partial\xi_1^{\alpha_1} \dots \partial\xi_n^{\alpha_n}$.

We shall often denote a generalized function by the symbol $f(\xi)$ in order to specify the argument of the basic functions. Let S^* be the space of generalized functions conjugate to the space S. The value of a functional $f \in S^*$ is frequently written in the form in the integral

$$(f,\ \varphi) = \int \overline{f(\xi)}\, \varphi(\xi)\, d\xi; \qquad (\bar{f},\ \varphi) = \overline{(f,\ \bar{\varphi})} = \int f(\xi)\, \varphi(\xi)\, d\xi.$$

The Fourier transformation of a generalized function $f \in S^*$ is expressed symbolically by

$$\tilde{f}(x) = \int f(\xi)\, e^{i\xi x}\, d\xi, \quad f(\xi) = \frac{1}{(2\pi)^n} \int \tilde{f}(x)\, e^{-i\xi x}\, dx.$$

Here by $\tilde{f}(x)$ we understand a functional $\tilde{f} \in S^*$ defined by the formula

$$(\tilde{f},\ \tilde{\varphi}) = (2\pi)^n (f,\ \varphi), \qquad \varphi \in S,$$

where $\tilde{\phi}(x)$ is the ordinary Fourier transform of the basic function $\phi(\xi)$; x is a point in the space $R_n^{(x)}$ of the real variables $x_1, \dots, x_n$.

Other necessary information on the theory of generalized functions can be found in the book of I. M. Gel′fand and G. E. Šilov, *Generalized functions and operations on them,* Vol. 1, Fizmatgiz, Moscow, 1962 (English translation, Academic Press, New York and London, 1964).

7. Supplementary remarks. Just as in the first part, we shall not use any special topics of the theory of analytic functions of one variable. The book by I. I. Privalov, *Introduction to the theory of functions of a complex variable,*

Gostehizdat, Moscow, 1954 (Russian), contains everything that we shall need here. The only exceptions are some facts from the theory of subharmonic functions. These can be obtained from another book by I. I. Privalov, *Subharmonic functions*, ONTI, Moscow, 1937 (Russian).

For reading sections of Chapter V dealing with the metric invariant under holomorphic mappings one will need some facts from tensor analysis and Riemannian geometry. These can be found in L. Eisenhart, *Riemannian geometry*, Princeton University Press, 1926 or K. Yano and S. Bochner, *Curvature and Betti numbers*, Princeton University Press, 1953.

In the sequel we shall use the concept of the convergence of multiple series. We note that in mathematical analysis one uses various definitions of the convergence of multiple series (see, for example, Osgood [1], (I), §13, Chapter I). We will say a multiple series R is convergent if every ordinary series converges that is obtained by arranging all the terms of the series R into one row in such a way that each term appears once and only once in this row.

CHAPTER I

APPROXIMATION OF FUNCTIONS AND DOMAINS

§1. DOMAINS OF CONVERGENCE

1. Construction of domains of convergence.[1)]

DEFINITION (domain of uniform convergence). A domain D over the space P^n is called a domain of uniform convergence of the first (second) kind if

1) there exists a sequence F of functions $\{f_j(z),\ j = 1, 2, \cdots\}$, holomorphic in the domain D, which converges uniformly in this domain to some holomorphic function (converges uniformly to infinity);

2) there is no domain $\widetilde{D} > D$ possessing the above property for the sequence F.

In such a case the domain D is said to be the *domain of uniform convergence* for the sequence F.

We remark that the domain D does not necessarily exhaust all the points z at which the sequence F converges (examples of such a sequence are given in §3.3, Chapter I, (I)).

We shall say that a sequence F converges uniformly to infinity in a domain D provided that, for any number $A > 0$ and for any domain $D_0 \ll D$, there exists a number N such that $|f_j(z)| > A$ for $z \in D_0$ and $j > N$.

The domain of a uniformly convergent sequence of holomorphic functions is constructed in the same way as the domain of holomorphy of a function. Consider a sequence F of functions $\{f_j(z),\ j = 1, 2, \cdots\}$, holomorphic in the neighborhood of some point $M_0 \in P^n$. By $S(M_0, r_0)$ we denote the largest elementary domain (of radius $r_0 > 0$ with center at the point M_0) in which all the functions $f_j(z)$ are holomorphic and the sequence F converges uniformly. Then we choose from the

1) These domains are discussed briefly in §12.7, Chapter II, (I).

domain S_0 a countable everywhere dense set of points $M_1, M_2, \cdots$, and determine the largest elementary neighborhood $S(M_k, r_k)$ to which all the functions $f_j(z)$ can be continued analytically and where a sequence F formed by these functions converges uniformly. Reasoning in the same way as in §9.1, Chapter II, (I), we obtain a sequence of elements $\{S_p\}$. For this sequence we define a set of constants ϵ_{pq} as follows: $\epsilon_{pq} = 1$ if $S_p \cap S_q \neq \emptyset$ and the functional elements belonging to each of the functions $f_j(z)$, $j = 1, 2, \cdots$, in the elements S_p and S_q coincide with each other in the intersection $S_p \cap S_q$; otherwise $\epsilon_{pq} = 0$. The domain D thus defined over the space P^n by means of the canonical covering $\{S_p, \epsilon_{pq}\}$ is also a domain of uniform convergence for the sequence F. It is obvious that the domain D is a subdomain of the domain Δ, the intersection (with respect to the point M_0) of the domains of holomorphy for the functions $f_j(z)$, $j = 1, 2, \cdots$. It is also clear that in constructing the domain D we can make use of the elementary domains of the first as well as the second kind.

A set F of functions holomorphic in some domain D over the space P^n forms there a normal family of the first (second) kind, if each of its infinite subsets contains a subsequence converging uniformly in the domain D to some holomorphic function (converging uniformly to infinity).

DEFINITION (normality domain). A domain D over the space P^n is called a normality domain of the first (second) kind if

1) there exists a set F of functions holomorphic in the domain D, which forms in this domain a normal family of the first (second) kind;

2) there is no domain $\widetilde{D} > D$ possessing the above property for the set F.

In such a case the domain D is said to be the *normality domain* for the family F.

If initially a normal family F is given in a neighborhood of some point M_0 in the space P^n, then the normality domain for this family can be constructed in the same way as in the previous case. The domain D thus constructed will be the subdomain of the domain Δ, namely the intersection (with respect to the point M_0) of the domains of holomorphy for the functions belonging to the family F.

All the theorems on the theory of normal families for holomorphic functions of one variable may be transferred without essential changes to the case of functions of several complex variables.[1] Among them we must note the following proposition

[1] See P. Montel, *Leçons sur les familles normales de fonctions analytiques et leurs applications*, Gauthier-Villars, Paris, 1927, pp. 193–203 of Russian translation. Hereafter we shall refer to this book as "Montel".

which will be used below in this section; it is an extended form of the Stieltjes theorem:

Let F be a sequence of holomorphic functions and let D be a normality domain of the first (second) kind for a family consisting of the functions of F. If there exists in the domain a point M_0 such that the sequence F converges in the neighborhood of this point, then the domain D coincides with the domain of uniform convergence of the first (second) kind for the sequence F.

This theorem can be proved, in general, in a manner which is analogous to the case of one variable.[1)]

Consider a series $\Sigma_{j=1}^{\infty} f_j(z)$ consisting of functions holomorphic in some domain D. We shall say that this series converges *normally* if the series $\Sigma_{j=1}^{\infty} \sup |f_j(D_0)|$ converges for any subdomain $D_0 \ll D$ of the domain D.

DEFINITION (domain of normal convergence). A domain D over the space P^n is called a domain of normal convergence if

1) there exists a series $\Sigma_{j=1}^{\infty} f_j(z)$ converging normally in the domain D;

2) there is no domain $\widetilde{D} > D$, in which this series also converges normally.[2)]

The domains considered in the present subsection are called, in their totality, the domains of convergence.

2. Properties of the domains of uniform convergence and the normality domains of the first kind.

THEOREM 1.1. *A normality domain of the first kind is always a domain of holomorphy.*

PROOF. Let D be a normality domain of the first kind for a family F of functions. Then, by definition, the domain D is a subdomain of the domain Δ which is the intersection of the domains of holomorphy for the functions belonging to the family F (thus, in the family F, we can certainly find functions which have different functional elements at different points of the domain D with identical coordinates). It follows from this that the domain D is holomorphically

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1) See Montel, pp. 34 and 199.

2) There is a close connection between the problems of normal convergence and the study of the space of holomorphic functions of several complex variables, as well as that of the bases in this space. The reader will find a presentation of these problems in an article by B. S. Mitjagin, *Approximative dimensionality and bases in kernel spaces*, Uspehi Mat. Nauk 16 (1961), no. 4 (100), 63–132. (Russian)

separable. Therefore our assertion will be proved (without reference to Oka's Theorem 11.3, (I)) if we can show that the domain D is holomorphically convex.

We shall prove the proposition for the case of a bounded domain. Let us start with the hypothesis that our assertion is not valid. Then there exists a subdomain D_0 of the domain D (where $D_0 \ll D$ and r is the minimal boundary distance of the domain D_0 in the domain D) and a sequence of points $M_k \in D$, $k = 1, 2, \cdots$, having no limit point in the domain D, (where $M_k \notin D_0$, $d_D(M_k) = r_k$, $\lim_{k\to\infty} r_k = 0$) such that for every function $f \in \mathfrak{O}_D$

$$f(M_k) \leq \sup |f(D_0)|. \tag{1.1}$$

Then, in view of Theorem 11.1, (I), all the functions $f \in \mathfrak{O}_D$ will be holomorphic in the polycylinder $S(M_k, r)$. We consider domains $D_0^{(\rho)}$ for values $\rho < r$. Since $D_0^{(0)} = D_0$, $D_0 \ll D$, we can choose a number ρ_0 so small that the domain $D_0^{(\rho 0)}$ will be a subdomain of D and $D_0^{(\rho 0)} \ll D$. In fact, if all domains $D_0^{(\rho 0)}$ do not have these properties, then there exists in each of the domains $D_0^{(\rho k)}$, $\lim_{k\to\infty} \rho_k = 0$, a sequence of points which has no limit point in the domain D. It is easy to show that in this case we can also construct in the domain D_0 a sequence of points having no limit point in D, which contradicts the hypothesis. Therefore we can assume that the domain $D_0^{(\rho 0)}$ is a subdomain of D and that $D_0^{(\rho 0)} \ll D$.

Since F is a normal family of the first kind, functions belonging to it will be bounded everywhere in the domain $D_0^{(\rho 0)} \ll D$.[1] Accordingly, there exists a number $R > 0$ such that for any function $\phi \in F$ we have the inequality

$$\sup |\phi(D_0^{(\rho 0)})| < R. \tag{1.2}$$

On the other hand in view of Theorem 11.1, (I), on the simultaneous continuation of a class of holomorphic functions, for all functions holomorphic in the domain D and, in particular, for the functions ϕ constituting the family F we have

$$\sup |\phi(S(M_k, \rho_0))| \leq \sup |\phi(D_0^{(\rho 0)})|. \tag{1.3}$$

From the relations (1.2) and (1.3), we see that

$$\sup |\phi(S(M_k, \rho_0))| < R.$$

Therefore the family F is bounded everywhere in the polycylinder $S(M_k, \rho_0)$ and accordingly it forms there a normal family of the first kind.[2] As $\lim_{r\to\infty} r_k = 0$, we can always assume that $r_k < \rho_0$; thus D is not a normality domain of the first

1) See Montel, p. 38 of Russian translation.

2) See Montel, pp. 28–32.

kind. In this way our hypothesis is refuted and Theorem 1.1 is proved.

In a completely analogous way, for a finite domain we can prove the following theorem.

THEOREM 1.2. *If D is a finite normality domain of the first kind for a family F of holomorphic functions, and if K is any class of functions, holomorphic in the domain D, which contains all functions of the family F, then D can be shown to be a K-convex domain.*

From this, of course, it again follows that D is a domain of holomorphy.

Analogous propositions could be obtained also for the domain of uniform convergence and for the domain of normal convergence of the series. The proof proceeds in just the same way as it did in Theorem 1.1. The proposition concerning the domain of uniform convergence can really be considered as a corollary of Theorem 1.1 and the Stieltjes theorem stated above. We leave it to the reader to deduce these propositions and restrict ourselves simply to the task of formulating them.

THEOREM 1.3. *The domain D of uniform convergence of the first kind is always a domain of holomorphy.*

If D is a finite domain of uniform convergence of the first kind for a sequence, and if K is an arbitrary class of functions, holomorphic in D, which contains all functions of the sequence F, then D can be shown to be a K-convex domain.

THEOREM 1.4. *The domain D of normal convergence for some series is always a domain of holomorphy.*

If D is a finite domain of normal convergence for some series, and if K is an arbitrary class of functions, holomorphic in the domain D, which contains all the functions forming terms of the series, then D is a K-convex domain.

From the last theorems follows the

COROLLARY. *If F is a normal family of the first kind in a finite domain D (a uniformly convergent sequence in D; a normally convergent series in D), and if K is an arbitrary class of functions, holomorphic in the domain D, which contains all the functions of F, then F is a normal family of the first kind in the hull $K(D)$ (a uniformly convergent sequence in the domain $K(D)$; a normally convergent series in the domain $K(D)$).*

We remark that the holomorphy hull $H(D)$ cannot be K-convex, since the domain of holomorphy of a function f should be K-convex only with respect to a class of functions, holomorphic in that domain, which includes the function f.

Therefore, the K-convex hull $K(D)$, in general, does not coincide with the holomorphy hull $H(D)$. The following theorem establishes a condition for such a coincidence.

THEOREM 1.5. *If each function holomorphic in a finite domain D can be represented as the limit of a uniformly convergent sequence consisting of functions from a class K, then the domains $H(D)$ and $K(D)$ coincide.*

PROOF. From the corollary formulated before the present theorem, we find that all functions holomorphic in the domain D are holomorphic also in the domain $K(D)$. Therefore $K(D) < H(D)$. On the other hand, since $K(D)$ is K-convex and finite (D is a finite domain, and so is $H(D)$; hence it follows that $K(D)$ is also a finite domain), $K(D)$ will be a domain of holomorphy by the corollary to Theorem 11.6, (I). Accordingly, $K(D) > H(D)$. Thus we have $K(D) = H(D)$.

COROLLARY. *If a class K in a finite domain D satisfies the conditions in the preceding theorem, then the domain D is a domain of holomorphy if and only if it is K-convex.*

Finally we shall state a proposition which is a converse of the first theorem in the present subsection.

THEOREM 1.6. *A finite domain D which is K-convex with respect to any class of functions holomorphic in the domain D is always a domain of uniform convergence of the first kind (a normality domain of the first kind; a domain of normal convergence) for some sequence F (family F) consisting of functions from the class K.*

PROOF. We consider the functions $\psi_\nu = [f_\nu]^{l_\nu}$ defined in the proof of Theorem 11.6, (I). Here $f_\nu, \psi_\nu \in K$. From the method of construction of these functions, it follows that $\lim_{\nu\to\infty} \psi_\nu = 0$ in D (see the inequality (2.25), (I)), and either the domain D coincides with the domain D_1 of uniform convergence for the sequence $\{\psi_\nu\}$ or it is a covering domain for D_1. In the second case we consider a sequence $\psi_1, \rho_1, \phi_1, \psi_2, \rho_2, \phi_2, \cdots$. Here ρ_ν and ϕ_ν are functions defined in the second part of the proof of Theorem 11.6, (I); clearly, they belong to the class K and in addition $\lim_{\nu\to\infty} \rho_\nu(z)\psi_\nu(z) = 0$ at every point $z \in D$. Then, as is obvious from the method of constructing this sequence, D is its domain of uniform convergence of the first kind. It follows from the generalized Stieltjes theorem formulated above that the domain D is also a normality domain of the first kind for some family of functions belonging to the class K.

For the case of normally convergent series the argument proceeds in an

analogous manner. We shall not discuss this question further.

3. Properties of the domains of uniform convergence and normality of the second kind. A solution of Julia's problem.

THEOREM 1.7. *The domains of uniform convergence and normality of the second kind are domains of meromorphy.*

The proof of this theorem is essentially analogous to the proof of Theorem 1.1. We shall not dwell on it. We have already shown (see §12.6, Chapter II, (I); in this connection, see also §14, Chapter III in the present volume) that a domain of meromorphy is always a domain of holomorphy. Therefore, we have from Theorem 1.7 the

COROLLARY. *The domains of uniform convergence and normality of the second kind are domains of holomorphy.*

It follows from Theorem 1.1 – 1.7 that the boundary points of the domains of convergence under consideration are boundary points of the domains of holomorphy (and the domains of meromorphy) for some of the analytic functions. In other words, they are singular (essentially singular) points of these functions.

Hence it follows that properties of the singular (essentially singular) points of the holomorphic functions which were established by the work of Hartogs and Levi and mentioned in §§9 and 12, Chapter II, (I), turn out to be properties of boundary points of the domains of convergence.

Formerly, G. Julia [1], (I) found all these properties for the normality domain by a direct method independent of the theory of singular and essentially singular points of holomorphic functions. Naturally there then arose the so-called "Julia's problem": namely, whether the normality domain is the domain of holomorphy (meromorphy). This question was answered affirmatively by H. Cartan and P. Thullen; we have mentioned their results above.

4. Applications to disk-shaped domains; n-circular domains over the space C^n. We have already seen (see §3.3, Chapter I, (I)) that the complete n-circular domain D with its center at a point $a = (a_1, \cdots, a_n)$ is the domain of uniform and absolute convergence for the power series,

$$\sum_k c_k (z-a)^k = f(z), \tag{1.4}$$

where, as usual, $c_k = c_{k_1 \cdots k_n}$, $(z-a)^k = (z-a_1)^{k_1} \cdots (z-a_n)^{k_n}$ and $k_1, \cdots, k_n = 0, 1, 2, \cdots$. It follows from Theorem 1.3 that the domain D is a domain of holomorphy. This implies, in particular, that the domain D is analytically convex

in the sense of Hartogs; if $n = 2$ and a hypersurface of the class $\mathcal{C}^2$ serves as the boundary of the domain D, then the domain can be shown to be logarithmically convex (see Theorem 13.9, (I)).

As we have already observed in §3.3, Chapter I, (I), for n-circular domains over the space C^n, we have the following proposition:

THEOREM 1.8. *A function $f(z)$ holomorphic in an n-circular domain D over the space C^n, to which its center*[1] *belongs, can be expressed in that domain by a uniformly convergent series* (1.4).

PROOF. We shall prove the theorem for the case of the space $C^2_{w,z}$. We put the origin of coordinates at the center of the domain D. Suppose that this domain is defined by means of the covering $\{S_i, \epsilon_{ij}\}$, where $S_i = \{|w - w_i| < R_i, |z - z_i| < R_i\}$ and, moreover, consider the domains D_r with $r > 1$ which are defined by the covering $\{S_i^{(r)}, \epsilon_{ij}\}$, where

$$S_i^{(r)} = \left\{ \left| w - \frac{w_i}{r} \right| < \frac{R_i}{r},\ \left| z - \frac{z_i}{r} \right| < \frac{R_i}{r} \right\}.$$

Suppose also that

$$F(w, z) = -\frac{1}{4\pi^2} \int_{C_r} \int_{\Gamma_r} \frac{f(ws, zt)}{(s-1)(t-1)}\, ds\, dt, \tag{1.4*}$$

where $C_r = \{|s| = r\}$ and $\Gamma_r = \{|t| = r\}$ are circles of radius r in the planes of the auxiliary complex variables s and t, respectively. The function $F(w, z)$ is holomorphic in the domain D_r (since the point $(wre^{i\theta}, zre^{i\phi}) \in D_r$ if and only if the point $(w, z) \in D$). It follows from (1.4*) that the functions $F(w, z)$ and $f(w, z)$ are identical in some neighborhood of the origin of coordinates. Accordingly, the function $f(w, z)$ coincides with the function $F(w, z)$ in the whole domain D_r and there we have

$$f(w, z) = -\frac{1}{4\pi^2} \int_{C_r} \int_{\Gamma_r} f(ws, zt) \frac{ds\, dt}{(s-1)(t-1)}$$
$$= \sum_{k,\, l=0}^{\infty} \varphi_{kl}(w, z), \tag{1.5}$$

where

$$\varphi_{kl}(w, z) = -\frac{1}{4\pi^2} \int_{C_r} \int_{\Gamma_r} \frac{f(ws, zt)}{s^{k+1} t^{l+1}}\, ds\, dt.$$

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[1] It is not a branch point if D is an interiorly-branching domain.

The series (1.5) converges uniformly in the domain D_r; the functions $\phi_{kl}(w, z)$ are holomorphic in this domain. In addition, we have

$$\varphi_{kl}(we^{i\theta}, ze^{i\varphi}) = -\frac{1}{4\pi^2} \int\limits_{C_r} \int\limits_{\Gamma_r} \frac{f(wse^{i\theta}, zte^{i\varphi})}{s^{k+1}t^{l+1}} ds\, dt = e^{i(k\theta+l\varphi)}\varphi_{kl}(w, z), \tag{1.6}$$

from which it follows that

$$\phi_{kl}(w, z) = a_{kl}w^k z^l \quad (a_{kl} \text{ are constants}). \tag{1.7}$$

In fact, in order to obtain equations (1.7), it is sufficient to form the derivatives

$$\frac{\partial^{m+n}}{\partial (e^{i\theta})^m \, \partial (e^{i\varphi})^n} \qquad (m = 0, 1, \ldots, k, \; n = 0, 1, \ldots, l)$$

of the function (1.6) and then to set $\theta = \phi = 0$. The equalities thus obtained readily give (1.7). In view of relations (1.7), the functions ϕ_{kl} are independent of r. Passing to the limit $r \longrightarrow 1$, we obtain the expression in question.

Theorem 1.8 implies that if the function $f(z)$ is holomorphic in some n-circular domain to which its center belongs, then it can be analytically continued to the smallest complete n-circular domain containing the given one. As was stated above, this domain will be analytically convex in the sense of Hartogs. Thus the incomplete n-circular domain to which its center belongs is not a domain of holomorphy.

Evidently the sum of the series (1.4) is always single-valued in the fundamental domain D. Therefore the n-circular domain to which its center (not being a branch point) belongs and which satisfies the condition of holomorphic separability is necessarily single-sheeted.

5. Semidisk domains over the space C^2. In the domain D of uniform and absolute convergence for the series (1.4), we have the equations

$$f(w, z) = \sum_{k=0}^{\infty} g_k(z)(w-a)^k, \tag{1.8}$$

where $g_k(z) = \sum_{l=0}^{\infty} c_{kl}(z-b)^l$ and $k = 0, 1, 2, \cdots$, in an appropriate neighborhood of the point $z = b$:

$$f(w, z) = \sum_{l=0}^{\infty} h_l(w)(z-b)^l, \tag{1.9}$$

where $h_l(w) = \sum_{k=0}^{\infty} c_{kl}(w-a)^k$ and $l = 0, 1, 2, \cdots$, in an appropriate neighborhood of the point $w = a$. Here we denote the variables by w and z and the coordinates of the center of the domain of uniform convergence for the series (1.4) by a and b.

In the way described in §1.1 we construct over the space C^2 a domain H of uniform convergence for the series (1.8). In considering the series (1.8) in the domain H, we shall understand by $g_k(z)$ a function which is obtained from the original function $g_k(z)$ by the process of analytic continuation. Generally speaking, $H > D$.[1] It should also be noticed that the domain H may be multiple-sheeted and the function $g_k(z)$ obtained by the analytic continuation may be many-valued in the domain $H \subset C^2$.

In a similar way we can construct the domain of uniform convergence for the series (1.9).

By Abel's first theorem in the theory of functions of a complex variable we can conclude that the domain H is a complete semidisk domain with the symmetry plane $w = a$. It follows from Theorem 1.3 that this domain H will be a domain of holomorphy.

If the domain H is defined by means of the covering $\{S_i(M_i, R_i), \epsilon_{ij}\}$ with $S_i(M_i, R_i) = \{|w - a_i| < R_i, |z - b_i| < R_i\}$, then the covering $\{s(b_i, R_i), \epsilon_{ij}\}$, where $s(b_i, R_i) = \{|z - b_i| < R_i\}$ and the constants ϵ_{ij} take on the same values as before, defines the *projection* H_z *of the domain* H *on the* z*-plane*. It can be shown that H_z is a domain of the z-plane, having (like the domain of holomorphy H) the property of holomorphic separability. If the domain H_z is single-sheeted, then H is also single-sheeted. The converse conclusion may be shown to be false; the domain H may be single-sheeted, while it has a many-sheeted projection on account of the fact that in the domain H there exist points which differ in their coordinates w but which have equal coordinates z.

Suppose that the complete semidisk domain H with the plane of symmetry $w = a$ is a domain of uniform convergence for the series (1.5). We take $a = 0$, so that this domain cuts from each plane $z = \text{const}$ (with which, in general, it has

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[1] Example. The function $1/(1-w)(1-z) = \sum_{k,l=0}^{\infty} w^k z^l$ in a bicylinder

$$\{|w| < 1, \ |z| < 1\}.$$

In the same bicylinder we can represent this function in the form $\sum_{k=0}^{\infty} w^k/(1-z)$. This series, however, converges in a larger domain, namely in the bicylindrical domain which is the product of the z-plane excluding $z = 1$ and the disk $|w| < 1$.

common points) the disk $|w| < R_z$. The quantity R_z is a function of z defined in the projection H_z of the domain H. At the boundary points of the domain H lying on the plane $z =$ const under discussion, $|w| = R_z$. Here R_z is called the *radius of uniform convergence* for the series $\sum_{k=0}^{\infty} g_k(z)w^k$ corresponding to the value z. Analogously, we can define the radius of uniform convergence R'_w for the series $\sum_{l=0}^{\infty} h_l(w)z^l$ corresponding to some value w.

If the domain $H \subset C^2$ and its boundary $\{|w| - R_z = 0\}$ is a hypersurface of the class $\mathcal{C}^2$, then in view of Theorem 12.7, (I), at points on this hypersurface the Levi determinant becomes

$$L(|w| - R_z) = -\frac{1}{4} R_z \left(\frac{1}{R_z} \frac{\partial^2 R_z}{\partial z \partial \bar{z}} - \frac{1}{R_z^2} \frac{\partial R_z}{\partial z} \frac{\partial R_z}{\partial \bar{z}} \right) = -\frac{1}{16} R_z \Delta (\ln R_z) \geqslant 0.$$

(This situation was mentioned before without derivation; see Theorem 13.10, (I).) From this follows the result $\Delta(\ln R_z) \leq 0$ for $z \in H_z$; thus the negative of the *logarithm of the radius of uniform convergence for the series* (1.5) *is a subharmonic function.* This is the so-called characteristic property of the radius of uniform convergence which was found previously by Hartogs. The quantity R_z is also called the *radius of holomorphy* for the sum of the series (1.8), i.e., the function $f(w, z)$.

For the semidisk domain the following proposition similar to Theorem 1.8 is valid:

THEOREM 1.9. *The function $f(w, z)$ holomorphic in a semidisk domain H over the space $C^2_{w,z}$, which has points*[1] *on its symmetry plane $w = a$, can be expressed in that domain by the series* (1.8).

PROOF. The proof is similar to that of Theorem 1.8. We put the origin[2] of coordinates on the symmetry plane of the domain H.

Suppose that the domain H is defined by the covering $\{S_i, \epsilon_{ij}\}$ with $S_i = \{|w - w_i| < R_i, \ |z - z_i| < R_i\}$.

Let $r > 1$ and H_r be the domain defined by the covering $\{S_i^{(r)}, \epsilon_{ij}\}$, where

$$S_i^{(r)} = \left\{ \left| w - \frac{w_i}{r} \right| < \frac{R_i}{r}, \ |z - z_i| < R_i \right\}.$$

We denote the circle $|t| = r$ by C_r and consider the function

1) These are not branch points if H is an interiorly-branching domain.

2) The origin of coordinates should not be placed at the branch point if H is an interiorly-branching domain.

$$F(w, z) = \frac{1}{2\pi i} \int\limits_{C_r} \frac{f(wt, z)}{t-1}\, dt. \tag{1.10}$$

This function is holomorphic in the domain H_r and (as is seen from equation (1.10)) is identical with the function $f(w, z)$ in the neighborhood of the origin of coordinates. But then, in the domain H_r we shall have

$$f(w, z) = \frac{1}{2\pi i} \int\limits_{C_r} \frac{f(wt, z)}{t-1}\, dt = \sum_{k=0}^{\infty} \varphi_k(w, z), \tag{1.11}$$

where

$$\varphi_k(w, z) = \frac{1}{2\pi i} \int\limits_{C_r} f(wt, z) \frac{dt}{t^{k+1}}.$$

The functions $\phi_k(w, z)$ are holomorphic in the domain H; in addition we have

$$\varphi_k(we^{i\theta}, z) = \frac{1}{2\pi i} \int\limits_{C_r} f(wte^{i\theta}, z) \frac{dt}{t^{k+1}} = e^{ik\theta}\varphi_k(w, k). \tag{1.12}$$

from which we find that

$$\phi_k(w, z) = w^k g_k(z). \tag{1.13}$$

To arrive at equation (1.13), we only need to take the derivatives $\partial^n/(\partial e^{i\theta})^n$ $(n = 0, 1, \cdots, k)$ of the functions (1.12) and then to set $\theta = 0$.

The function $g_k(z) = \phi_k(w, z)/w^k$ is defined as the ratio of two holomorphic functions in the domain H_r and, accordingly, it is a meromorphic function there. In the neighborhood of the origin it depends only on the variable z; the same must be true in the whole domain H_r. By definition this function may have singularities only on the plane $w = 0$. In view of Theorem 9.1, (I), on the continuous distribution of singularities of analytic functions, either the whole of this plane (as long as it belongs to the domain of meromorphy of a function) consists of the singular points of the function or else it nowhere contains such points. In our case the second possibility occurs, since the function $g_k(z)$ is holomorphic in the neighborhood of the point $(0, 0)$. Thus it is shown that $g_k(z)$ is a holomorphic function in the whole domain H_r.

By definition, the function $\phi_k(w, z)$ is independent of r. Taking the limit $r \longrightarrow 1$ in equation (1.11) we obtain the expansion that was involved in the assertion of the theorem.

Finally we remark that the expansion (1.8) is always unique and independent

of the method used to obtain it. From equation (1.8) it follows that

$$'g_k(z) = \frac{1}{k!}\frac{\partial^k f(w, z)}{\partial w^k}\Big|_{w=0}.$$

Theorem 1.9 implies that if the function $f(w, z)$ is holomorphic in some semidisk domain H having points on its plane of symmetry, then it can be analytically continued to the smallest complete semidisk domain which has the same symmetry plane, has a boundary convex in the sense of Hartogs and contains the domain H. Therefore the incomplete semidisk domains having points on the symmetry plane are not domains of holomorphy.

6. Disk domains over the space C^2. In the domain D of uniform (and absolute) convergence for the series (1.4), the following equality holds:

$$f(w, z) = \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} a_{k-l,l}(w-a)^{k-l}(z-b)^l\right). \tag{1.14}$$

The series on the right-hand side of (1.14) is called the *diagonal series*. In the manner described in §1.1 we can construct the domain K of uniform convergence for the series (1.14). Generally speaking, $K \supset D$.

By Abel's first theorem on the theory of functions of a complex variable we can conclude that the domain K is a complete disk domain with center at the point (a, b). Such a domain is always star-shaped and homeomorphic to a hyperball. From Theorem 1.3 it follows that K is a domain of holomorphy. By the property of holomorphic separability and by Theorem 6.1, (I), on monodromy, the domain K is single-sheeted.

For the study of the domain of uniform convergence for the series (1.14), it is convenient to put $z - b = (w - a)t$; then the series takes the form $\Sigma_{k=0}^{\infty} g_k(t)(w-a)^k$, where $g_k(t)$ is a polynomial of order k in t. We obtain a series of the form (1.8) and the domain K goes over into the complete semidisk domain H of uniform convergence for this series. It is single-sheeted, just like the domain K (as stems from the fact that the polynomial $g_k(t)$ remains single-valued for any analytic continuation). By repeating the argument for the domain H we can conclude that the boundary of the domain K is defined outside the plane $w = a$ by the equation $|w - a| = R(t)$ with $t = (z - b)/(w - a)$ and outside the plane $z = b$ by the equation $|z - b| = R^*(t^*)$ with $t^* = (w - a)/(z - b)$. Here $\ln R(t)$ and $\ln R^*(t^*)$ are superharmonic functions (this situation was indicated in Chapter II, (I), but in another form and without proof; see Theorem 13.10, (I)).

Finally we note that for the disk domain the following proposition similar to

Theorems 1.8 and 1.9 is valid:

THEOREM 1.10. *Any function $f(w, z)$ holomorphic in a disk domain K containing its center can be expressed in that domain K by the uniformly and absolutely convergent diagonal series* (1.14).

From this theorem and the properties of the domain of uniform convergence for the series (1.14) it follows that the incomplete disk domains to which its center belongs are not domains of holomorphy.

§2. APPROXIMATION BY MEANS OF FUNCTIONS BELONGING TO A COMPLETE FAMILY

1. Formulation of the fundamental theorem. The present section is devoted to the proof of a theorem of K. Oka which forms the basis of many results in the theory of approximation.[1)]

THEOREM 2.1. *Let D be a domain of the space C^n and let Φ be some complete family of functions holomorphic in that domain D. Every function $F(z)$ holomorphic in the domain D can be represented as the sum of a series converging uniformly in D and consisting of functions which belong to the family Φ, if and only if the holomorphy hull $H(D)$ of this domain D is convex relative to the family Φ.*

PROOF. The complete family of functions is always a class (see §11.1, Chapter II, (I)). Hence the necessity of the condition indicated in the theorem follows immediately from Theorem 1.5. Theorem 2.1 will be completely proved if we can establish its sufficiency for the case when D is a domain of holomorphy. Then $H(D) = D$. We will divide this part of the proof into several steps.

2. Sufficiency proof in the case when Φ is a family of polynomials or rational functions. We suppose, for definiteness, that Φ is the family of all polynomials. The case when Φ is the family of all rational functions holomorphic in the domain D can be treated in exactly the same way.

Take some domain $D_0 \Subset D$. In view of the Φ-convexity of the domain D the inclusion relation $D_0 \subset (\hat{D}_0)_\Phi \Subset D$ holds (see §11.2, Chapter II, (I)); put $\delta_0 = d_D[(\hat{D}_0)_\Phi]$. Next take a domain D_1 such that $(\hat{D}_0)_\Phi \Subset D_1 \Subset D$, and so forth. In this way we construct a sequence of domains $D_p \Subset D$, $p = 0, 1, 2, \cdots$, possessing the following properties: 1) $D_p \subset (\hat{D}_p)_\Phi \Subset D_{p+1}$; 2) $\delta_p = d_D[(\hat{D}_p)_\Phi] \longrightarrow 0$ as

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1) Oka [1], (I). See another formulation of Theorem 2.1 in subsection 1 of the next section.

$p \to \infty$. Here, in general, the distances are taken in the chordal metric (see §5, Chapter I, (I)); if D is a bounded domain, it is convenient to take them in the metric introduced in §1.1, Chapter II, (I). In the latter case $d_D[(\hat{D}_p)_\Phi] = d_D(D_p)$ (by Theorems 11.6, (I) and 11.1, (I)).

Finally, let $\Sigma_p = \{z \in D,\ d_D(z) = \delta_p/2\}$. Then, at every point $M \in \Sigma_p$, one can find a polynomial $\phi(z)$ such that

$$|\phi(M)| > 1, \tag{1.15}$$

and at the same time

$$\sup|\phi(D_p)| < 1. \tag{1.16}$$

By continuity considerations it further follows that the inequality (1.15) still holds in some neighborhood S of this point M.

Evidently Σ_p is a compact set; therefore, by Borel's lemma, it can be covered by a finite set of similar neighborhoods S_j of points $M_j \in \Sigma_p$ $(j = 1, \cdots, N)$. Let $\phi_j^{(p)}(z)$ be corresponding polynomials which satisfy the conditions

$$|\varphi_j^{(p)}(z)|_{z \in S_j} > 1, \quad \sup|\varphi_j^{(p)}(D_p)| < 1.$$

We now consider a set of those points of the domain D for which

$$|\varphi_j^{(p)}(z)| < 1, \qquad j = 1, \ldots, N.$$

Among the connected components of this set one can find a domain which contains the domain D_p inside it. We denote it by Δ_p; we may always assume that Δ_p is a Weil domain.[1] It is evident that its boundary consists of hypersurfaces $|\phi_j^{(p)}(z)| = 1$, $j = 1, \cdots, N$ and, in view of the inequalities (1.15) and (1.16), it is situated strictly outside the domain D_p and inside the domain D. Then, by Theorem 22.4, (I) due to Weil, every function $F(z)$ holomorphic in the domain D can be represented in the domain D_p by an absolutely and uniformly convergent series of polynomials.

Let $F(z)$ be some function holomorphic in the domain D. We take a sequence of numbers ϵ_p $(p = 0, 1, 2, \cdots)$ decreasing monotonically as $p \to \infty$. Then by

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[1] For the original choice it may be that the polynomials $\phi_j^{(p)}(z)$ do not satisfy all the conditions in §22.1, (I), and then the domain Δ_p will not be a Weil domain. However, making use of the available arbitrariness in the choice of the points M_j and of the polynomials $\phi_j^{(p)}(z)$ themselves, we can ensure that all the requirements imposed on the Weil domain are satisfied.

what has been proved, there exists a polynomial $P_0(z)$ such that in the domain D_0

$$|F_0(z)| < \epsilon_0,$$

where $F_0(z) = F(z) - P_0(z)$. The function $F_0(z)$ is holomorphic in the domain D. Therefore there exists a polynomial $P_1(z)$ such that in the domain D_1

$$|F_1(z)| < \epsilon_1,$$

where $F_1(z) = F_0(z) - P_1(z) = F(z) - \Sigma_{k=0}^{1} P_k(z)$. Reasoning in this way, we define polynomials $P_k(z)$, $k = 2, 3, \cdots$, such that in the domain D_p

$$\left| F(z) - \sum_{k=0}^{p} P_k(z) \right| < \varepsilon_p.$$

For any domain $\mathfrak{D} \Subset D$ one can find a number p such that $\mathfrak{D} \subset D_p$. Therefore the series of polynomials $\Sigma_{k=0}^{\infty} P_k(z)$ which we have obtained converges absolutely and uniformly in the domain D to the function $F(z)$. With this our assertion is proved.

To prove Theorem 2.1 in the general case we have to introduce a series of auxiliary concepts. We shall achieve this by first deducing several lemmas.

3. Solution of Cousin's first problem for Weil polynomial sets. In the space C_z^n of the variables $z_1, \cdots, z_n$ we consider an open set

$$\Delta = \{z_i \in D_i,\ P_j(z) \in E_j,\ i = 1, \ldots, n;\ j = 1, \ldots, N\}.$$

Here D_i is a domain on the plane of the variable z_i, while E_j is a domain on the plane of the new variable w_j and $P_j(z)$ is a polynomial. Such an open set will be called, by the analogy of a Weil polynomial polyhedron (see §22, Chapter IV, (I)), a Weil polynomial set. In the sequel we assume that in the representation (1.17) of the set Δ none of the polynomials $P_1, \cdots, P_N$ can be dropped. In this case the number N is called the *order of the set* Δ.[1] Thus the Weil polynomial set of order zero is a polycylindrical domain.

Next, in the space $C_{z,w}^{n+N}$ of the variables $z_1, \cdots, z_n, w_1, \cdots, w_N$ we consider a set

$$\Sigma = \{z \subset \Delta,\ w_j = P_j(z),\ j = 1, \ldots, N\}.$$

Evidently $\Sigma \subset C = \{z_i \in D_i,\ w_j \in E_j,\ i = 1, \cdots, n;\ j = 1, \cdots, N\} \subset C_{z,w}^{n+N}$.

[1] This name will be retained when the P_j are not polynomials but are, instead, arbitrary holomorphic functions.

With each function $f(z)$ given on the set Δ, we put into correspondence a function on the set Σ which assumes at a point $M(z, w) \in \Sigma$ the value $f(z)$. We shall denote this function by $f(M)$.

Finally, let C_0 be some open set, strictly included in a polycylindrical domain C, and let $\Sigma_0 = C_0 \cap \Sigma$.

We consider the following problems:

Cousin's first problem for a Weil polynomial set Δ. To each point $M \in \Delta$ there corresponds some neighborhood $V_M \subset \Delta$ and a function f_M meromorphic in this neighborhood. If $V_{M_1} \cap V_{M_2} \neq \emptyset$, then in this intersection the functions f_{M_1} and f_{M_2} are equivalent, relative to subtraction. The problem is to determine on the set Δ a meromorphic function F which is equivalent, relative to subtraction at every point $M \in \Delta$, to the given function f_M.

By analogy we can formulate Cousin's first problem for the interior of the set Δ. In this case the function f_M corresponds to all the points of the set Δ, and the function F is determined on an arbitrary neighborhood of a set $\mathfrak{D}_0 \Subset \Delta$.

First auxiliary problem. It is required to determine on the set C_0 a holomorphic function $F(z, w)$ which coincides with the function $f(M)$ on the set Σ_0. The function $f(z)$ here is assumed to be holomorphic on the set Δ.

In the domain D_1, consider a rectifiable line L joining two points z'_1, $z''_1 \in D_1$. Let T be the collection of those points of the set for which $z_1 \in \overline{L}$, and let $T_0 = T \cap \overline{\mathfrak{D}}_0$, where $\mathfrak{D}_0 \Subset \Delta$. In a neighborhood $U_T \subset \Delta$ of the set T we are given a holomorphic function $f(z)$.

Second auxiliary problem. To find on the set $\mathfrak{D}_0$ a function $\phi(z)$ holomorphic everywhere except at points of the set T_0 and possessing the following properties:

a) On the set T_0 the function $\phi(z)$ must undergo a discontinuous jump equal to $f(z)$ (on crossing to the side of T_0 corresponding to the values of z_1 which lie on the left of the line L running from z'_1 to z''_1). Both branches of the function $\phi(z)$ should be analytically continuable through T_0; points corresponding to the values $z_1 = z'_1$, z''_1 are exceptional.

b) In the neighborhood of the point z of the set T_0, corresponding to the value $z_1 = z'_1$ (or to $z_1 = z''_1$ respectively), the analytic continuation of the function $\phi(z) - (1/2\pi i) f(z) \ln(z'_1 - z_1)$ must remain holomorphic.

The number N (i.e., the difference of the complex dimensions of the spaces C^{n+N} and Σ) will be called the *order* of the first and second auxiliary problems, the order of Cousin's first problem formulated for the interior of the polynomial

set Δ. Cousin's Theorem 25.1, (I) establishes that Cousin's first problem of order zero is solvable.

First of all we shall prove that the following lemma holds:

LEMMA 1. *If the first auxiliary problems and Cousin's first problems of order less than N are solved, then the first auxiliary problem of order N is solvable.*

PROOF. In the space C^{n+1} of the variables $z_1, \cdots, z_n, w_N$ we consider an open set $G = \{z_i \in D_i, w_N \in E_N, P_j(z) \in E_j;\ i = 1, \cdots, n;\ \ j = 1, \cdots, N-1\}$ and a set $S = \{z \in \Delta, w_N = P_N(z)\}$.[1)]

To each point $M \in G$ we associate as its neighborhood an equilateral polycylinder V_M with its center at this point. We choose its radius so small that if the point $M \notin S$, then $V_M \cap S = \emptyset$; if the point $M \in S$, then $\tau V_M \subset \Delta$. Here τ is the projection $(z, w_N) \longrightarrow z$.

Let $f(z)$ be a function holomorphic on the set Δ. In each polycylinder V_M we are given a function $h_M(z, w)$ by the following equations:

$$h_M(z, w_N) = \begin{cases} 0, & \text{if the point } M(z, w_N) \notin S; \\ \dfrac{f(z)}{w_N - P_N(z)}, & \text{if the point } M(z, w_N) \in S. \end{cases}$$

The neighborhoods V_M and the meromorphic functions $h_M(z, w_N)$ thus defined satisfy the conditions of Cousin's first problem. The set G has an order not exceeding the number $N-1$. Hence, because of our hypothesis, on each open set $G_0 \Subset G$ there exists a meromorphic function $H(z, w_N)$ which is equivalent, in all neighborhoods V_M, to the corresponding functions $h_M(z, w_N)$.

Next we put

$$\varphi(z, w_N) = H(z, w_N)\,[w_N - P_N(z)].$$

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1) For clarity it is recommended that the reader visualize step by step the corresponding construction for the real variables x_1, x_2. Suppose, for example, that $D_1 = \{0 < x_1 < 3\}$, $D_2 = \{0 < x_2 < 3\}$; $P_1 = x_1 + x_2$, $P_2 = x_1 x_2$; $E_1 = \{0 < x_1 + x_2 < 3\}$, $E_2 = \{0 < x_1 x_2 < 1\}$. If we replace w_N (here $N = 2$) by a variable y, then in the space of variables x_1, x_2, y the domain G will correspond to a right prism based on a triangle with vertices at points $(0, 0, 0)$, $(3, 0, 0)$, $(0, 3, 0)$ and having height equal to unity. Here Δ is a portion of this triangle containing the origin and cut from it by the hyperbola $x_1 x_2 = 1$ and S is a piece of the paraboloid $y = x_1 x_2$ lying in the prism G. The domain Δ is the projection of S on the x_1, x_2-plane.

This function is holomorphic on the set G_0 and, moreover, on the set $S_0 = S \cap G_0$ we have

$$\phi(z, P_N(z)) = f(z). \tag{1.18}$$

Let us also set $G_0 = \{z_i \in D_i^{(0)},\ w_N \in E_N^{(0)},\ P_j(z) \in E_j^{(0)};\ i = 1, \cdots, n;$ $j = 1, \cdots, N-1\}$. Here $D_i^{(0)}$, $E_j^{(0)}$ are plane domains lying entirely inside the domains D_i, E_j. These are chosen in such a way that $C_0 \subset (D_1^{(0)} \times \cdots \times D_n^{(0)} \times E_1^{(0)} \times \cdots \times E_N^{(0)})$ (see the definition of the first auxiliary problem).

We now construct a function $F(z, w)$ on the set C_0 satisfying the condition

$$F(z, P_1, \cdots, P_{N-1}, w_N) = \phi(z, w_N).$$

The problem of constructing such a function constitutes the first auxiliary problem of order less than $N-1$ (since the problem is to find a function assuming given values on the set $w_j = P_j(z)$, $j = 1, \cdots, N-1$; the reader should recall the meaning of the order of the problem which was stated in parenthesis at the time of its definition).

Then, in view of equation (1.18), we have

$$F|_{\Sigma_0} = F(z, P_1, \ldots, P_{N-1}, P_N) = \varphi(z, P_N) = f(z).$$

Thus we have obtained the desired solution. Lemma 1 is proved.

From our argument the following corollary also results:

COROLLARY. *The first auxiliary problem of first order is always solvable.*

LEMMA 2. *If all the first auxiliary problems of order less than or equal to N are solved, then the second axuiliary problems of these orders are also solvable.*

PROOF. Consider an open set $\Delta_0 = \{z_i \in D_i^{(0)};\ P_j(z) \in E_j^{(0)};\ i = 1, \cdots, n;$ $j = 1, \cdots, N\}$, where the domains $D_i^{(0)} \Subset D_i$ and $E_j^{(0)} \Subset E_j$. Whatever the open set $\mathfrak{D}_1 \Subset \Delta$ may be, the domains $D_i^{(0)}$ and $E_j^{(0)}$ can be taken so that $\mathfrak{D}_1 \Subset \Delta_0$. Therefore for the proof of the lemma it is sufficient for us to consider sets Δ_0 of the above kind.

We choose a neighborhood U_T of the set T such that for points $(z_1, \cdots, z_n) \in U_T$ the corresponding coordinates $z_1 \in \widetilde{L}$. Here $\widetilde{L} \Subset D_1$ is some domain of the z_1-plane which contains the line L. In addition, we take on the z_1-plane a domain $\widetilde{L}_1$ such that: 1) $\widetilde{L} \Subset \widetilde{L}_1 \Subset D_1$; 2) at points $(z_1, \cdots, z_n) \in \Delta$, for which $z_1 \in \widetilde{L}_1$, the function $f(z)$ is holomorphic.

Consider the open set B which is obtained in place of the set Δ if, in the definition, we replace the domain D_1 by the domain $\widetilde{L}_1$, and the open set B_0

which is obtained in place of the set Δ_0 by replacing the domain $D_1^{(0)}$ by the domain $\widetilde{L}$. Suppose that sets $\gamma, \gamma_0, \sigma, \sigma_0$ play for the sets B and B_0 the same roles as the sets C, C_0, Σ, Σ_0 play for the sets Δ and Δ_0.

We shall determine a function $F(z, w)$ which is holomorphic on the set γ_0 and equal to a given function $f(z)$ on the set σ_0. To this end we have only to solve the first auxiliary problem of order at most N, but this is possible in view of our assumption.

We now consider the function

$$\Phi(z, w) = \frac{1}{2\pi i} \int_L \frac{F(t, z_2, \ldots, z_n, w)}{t - z_1} dt \tag{1.19}$$

(here the integral is taken from the point z_1' to the point z_1''). Properties of the function Φ may be studied in a manner which is completely analogous to the case of the integral (5.9), (I) (in Chapter V, (I)). Such an investigation then leads us to similar results. The function Φ turns out to be holomorphic on the set $C_0 - \lambda_0$ (where λ_0 is a collection of those points of the set C_0 for which $z_1 \in \overline{L}$); moreover, it will be holomorphic on the set which is obtained from the set C_0 by replacing, in the definition, the domain $D_1^{(0)}$ by the entire z_1-plane. Points of the set λ_0 are again exceptional. It further appears that the function Φ has the properties a) and b) stated in the formulation of the first auxiliary problem if we replace the sets Δ_0 and T_0 by C_0 and λ_0, and the functions f and ϕ by F and Φ, respectively.

We consider values of the function Φ at points of the set $\Sigma_0 = \{z_i \in \Delta_0, w_j = P_j(z);\ i = 1, \cdots, n;\ j = 1, \cdots, N\}$ and put

$$\Phi(z, P_1, \cdots, P_N) = \phi(z).$$

Thus on the set Δ_0 we have defined the function $\phi(z)$ holomorphic everywhere except at points of the set T_0 (corresponding to the set $\lambda_0 \subset C_0$). The behavior of the function ϕ on the set T_0 is determined by that of the function Φ on the set λ_0; the jump of the function ϕ at points of the set T_0 will be equal to

$$F(z, P_1, \cdots, P_N) = f(z).$$

Therefore the function $\phi(z)$ constructed by us is the desired one. Lemma 2 is proved.

LEMMA 3. *Cousin's first problem of any order is solvable.*

PROOF. The proof can be reduced to arguments which have been already

carried out. In the corollary of Lemma 1 we have shown that the first auxiliary problem of the first order is solvable. We shall prove the solvability of Cousin's first problem of first order. For this purpose we may subdivide the domain D_i into smaller domains $d_a^{(i)}$, which will be substituted for the squares $q_a^{(i)}$ in the proof of Cousin's first theorem (25.1, (I)). Instead of the domain $q_{a_1}^{(1)} \times \cdots \times q_{a_n}^{(n)}$ we need to consider the *part* of the domain $d_{a_1}^{(1)} \times \cdots \times d_{a_n}^{(n)}$ belonging to the open set Δ. As for the rest we have only to repeat Cousin's argument (using, of course, the integral (1.19) instead of the integral (5.9), (I)). Thus we have established that Cousin's first problem of first order is solvable. Hence, by Lemma 1, it follows that the first auxiliary problem of second order is solvable, and by Lemma 2 that the second auxiliary problem of second order is solvable. Then, again repeating Cousin's argument (with the above-mentioned changes), we can prove the solvability of Cousin's first problem of second order, and so forth. Lemma 3 is proved.

REMARK. In order to solve Cousin's first problem for the entire set Δ (not for its part $\Delta_0 \Subset \Delta$), it is necessary to apply, in addition to the above construction process, the limiting transition which was used at the end of the proof of Cousin's first theorem. However, we shall not dwell on this.

4. **π_1-hulls.** We shall say that the Weil polynomial sets of the space C_z^n defined by the condition

$$\Delta = \{ |z_i| < r_i; \ \ |P_j(z)| < 1; \ \ i = 1, \dots, n; \ j = 1, \dots, N \}$$

constitute the *class* π_0. Here r_i are certain positive numbers, and the $P_j(z)$ are polynomials as before. We agree to say that a closed set $F \subset C_z^n$ belongs to the *class* π_1 if there exists a sequence of sets $F_\nu \in \pi_0$, $F_\nu \supset F_{\nu+1}$, $\nu = 0, 1, 2, \cdots$, such that $\lim_{\nu \to \infty} F_\nu = F$.

Let $E \subset C^n$ be a bounded set. The common part $\mathfrak{S}$ of all of those sets of the class π_1 which contain the set E is called the π_1-hull of the set E. In this case we write $\pi_1(E) = \mathfrak{S}$.

LEMMA 4. $\pi_1(E) = \mathfrak{S} \in \pi_1$ *(i.e., $\mathfrak{S}$ is the smallest set of the class π_1 which contains the set E).*

PROOF. Let $\overline{E} \subset C = \{|z_i| < R_0, \ i = 1, \cdots, n\}$, and let F be a set of the class π_1 such that $E \subset F \subset C$. Then $\mathfrak{S}$ is the common part of all similar sets $F \in \pi_1$.

By ρ_0 we denote the smallest boundary distance of the set $\mathfrak{S}$ in the polycylinder C. Such a number $\rho_0 > 0$ certainly exists, since among the sets F we have

polycylinders $\{|z_i| < R,\ i = 1, \cdots, n\}$ of radius $R < R_0$.

For every positive number $\rho < \rho_0$ we form a closed set A_ρ consisting of those points of the closed polycylinder $\overline{C}$ which are separated from the set $\mathfrak{S}$ by a distance greater than or equal to ρ.

We consider a certain point $M \in A_\rho$. Since the point $M \notin \mathfrak{S}$, it fails to belong to certain sets of F and consequently to certain sets of the class π_0 which approximate these sets F. Therefore there exists a polynomial $P^{(\rho)}(z)$ and a polycylinder $S(M, r)$ of radius r such that

$$|P^{(\rho)}(z)| > 1 \quad \text{for} \quad z \in S(M, r),$$
$$|P^{(\rho)}(z)| < 1 \quad \text{for} \quad z \in \mathfrak{S}.$$

By Borel's lemma the set A_ρ can be covered by a finite set of similar polycylinders $S(M_j, r_j)$, $j = 1, \cdots, N$; let $P_j^{(\rho)}(z)$ be corresponding polynomials.

We form a set $\Delta_\rho = \{z \in C,\ |P_j^{(\rho)}(z)| < 1,\ j = 1, \cdots, N\}$. Evidently $\Delta_\rho \in \pi_0$, $\mathfrak{S} \subset \Delta_\rho$. By constructing a sequence of similar sets Δ_{ρ_n}, where $\lim_{n\to\infty}\rho_n = 0$, we complete the proof of our lemma.

We consider a family $\{X_t\}$ of $(n-1)$-complex-dimensional analytic surfaces $X_t = \{f(z, t) = 0,\ 0 \le t \le 1,\ z \in U\}$. Here U is a bounded domain of the space C_z^n and $f(z, t)$ is an analytic function of real variables t, $x_i = \operatorname{Re} z_i$, $y_i = \operatorname{Im} z_i$ $(i = 1, \cdots, n)$ in the closed domain $\overline{U} \times \{0 \le t \le 1\}$ and, moreover, it is a holomorphic function, distinct from the identically vanishing function, of the variables z_i $(i = 1, \cdots, n)$ in the domain $U \subset C_z^n$ for every value of the variable t in the closed interval $\{0 \le t \le 1\}$.

LEMMA 5. *There can not exist a family of analytic surfaces $\{X_t\}$ which has the following properties*:

1) *The boundaries of all the surfaces X_t do not enter some neighborhood $\tilde{\mathfrak{S}}$ of the set $\mathfrak{S}$.*

2) *The surfaces X_t do not possess points in common with the set $\overline{E}$.*

3) *The surface X_0 passes through some point of the set $\mathfrak{S}$, while the surface X_1 lies entirely outside the set $\tilde{\mathfrak{S}}$.*

PROOF. Suppose that, contrary to our assertion, there exists a family of analytic surfaces $\{X_t\}$ with the three above-mentioned properties. Then there exists a domain T of the plane of the complex variable t, belonging to the class π_0, such that the family of surfaces $X_t = \{f(z, t) = 0,\ t \in T\}$ also has the properties 1), 2), 3). Here it is assumed that the domain T contains the segment $0 \le t \le 1$

of the real axis; it may be possible that the sets U and $\tilde{\mathfrak{S}}$ could be reduced somewhat in going over to the domain T; the existence of such a domain T follows from continuity considerations.

As the set $\mathfrak{S} \in \pi_1$, we can choose its neighborhood $\tilde{\mathfrak{S}} \in \pi_0$. Then the open set $\tilde{\mathfrak{S}} \times T$ also belongs to the class π_0 and Lemma 3 can be applied to it.

From this lemma it results that in the domain $\tilde{\mathfrak{S}} \times T$ there exists a meromorphic function $g(z, t)$ holomorphic at points where $f(z, t) \neq 0$ and equivalent to to the function $[f(z, t)]^{-1}$ in the neighborhoods of the points at which $f(z, t) = 0$. These last neighborhoods are chosen so that the conditions of the formulation of Cousin's first problem are fulfilled.

Let $t = a$, where $0 \leq a < 1$, be the largest value of the parameter t, for which the surface X_t still has points in common with the set $\mathfrak{S}$. Such a value of the parameter t exists, since $\mathfrak{S}$ is a closed set. Take a point $z^{(0)} \in (X_a \cap \mathfrak{S})$. Then the function $g(z^{(0)}, t) \longrightarrow \infty$ as $t \longrightarrow a$ along the segment $a < t \leq 1$, since the point $(z^{(0)}, t)$ is a pole of the function g. On the other hand, this function is holomorphic and accordingly is bounded at all the points of the set $\overline{E} \times \{0 \leq t \leq 1\}$ (by the second property of the family of surfaces $\{X_t\}$). Therefore, on the segment $a < t \leq 1$, one can find a value $t = \beta$ in any vicinity of the point $t = a$ such that

$$\sup |g(\overline{E}, \beta)| < |g(z^{(0)}, \beta)|. \tag{1.20}$$

The function $g(z, \beta)$ is holomorphic at every point of the closed set $\mathfrak{S} \in \pi_1$. Therefore this function can be expressed on the set $\mathfrak{S}$ by a uniformly convergent series of polynomials. To this end it is necessary to find in the sequence of sets from the class π_0 which approximates the set $\mathfrak{S}$ a set $\mathfrak{S}_1 \in \pi_0$ such that the function $g(z, \beta)$ remains holomorphic at $z \in \mathfrak{S}_1$. Further it is necessary to recall Weil's Theorem 22.4, (I). As a result we find a polynomial $\Phi(z)$ for which, by the inequality (1.20), we have

$$\sup |\Phi(\overline{E})| < 1 < |\Phi(z^{(0)})| \tag{1.21}$$

(it will be necessary to multiply the ~~polynomial~~ by an appropriate constant factor in order to get the unity between the two inequality signs). The inequality (1.21) contradicts the assumption that $\mathfrak{S}$ is the smallest set of the class π_1 which includes the set E. Lemma 5 is proved.

5. A lemma on subharmonic functions. Now we consider the closed set $\overline{\Delta} = \{ |f_j(z)| \leq 1, \ j = 1, \cdots, N\} \subset C_z^n$, whose connected components are closed analytic polyhedra (see §22, Chapter IV, (I)). Here it is assumed that all the

functions $f_j(z)$ are holomorphic in some domain $D \subset C_z^n$ and that $\bar{\Delta} \subset D \subset \{|z_i| < r_i, i = 1, \cdots, n\}$, where r_i are certain positive numbers.

In the same way as above, in the space $C_{z,w}^{n+N}$ of the variables $z_1, \cdots, z_n, w_1, \cdots, w_N$ we consider the closed set $\Sigma = \{w_j = f_j(z),\ z \in \bar{\Delta},\ j = 1, \cdots, N\}$. Evidently $\Sigma \subset \bar{C}$, where $C = \{|z_i| < r_i, |w_j| < 1,\ i = 1, \cdots, n;\ j = 1, \cdots, N\}$.

Consider the hull $\pi_1(\Sigma) = S$. It is obvious that $S \subset \bar{C}$, since $\Sigma \subset \bar{C}$ and $\bar{C} \in \pi_0$. A fundamental result of K. Oka [1], (I), from which the assertion of Theorem 2.1 easily follows, is the proof of the equality $\Sigma = S$ (Lemma 8). Thus the set Σ is itself in π_1. In order to establish this proposition we must first prove two additional lemmas (6 and 7). We shall now begin by deducing these last two lemmas.

Let σ be the projection of the set S onto the space C_z^n, and let s be the projection of the set S onto the plane $z_1 = z_1^{(0)}$. We consider the sections of the sets S, σ in the domain D by the plane $z_1 = z_1^0$; we denote by $S_{z_1^0}$, $\sigma_{z_1^0}$, and $D_{z_1^0}$ the projections of these sections onto the space C^{n+N-1} of the variables $z_2, \cdots, z_n, w$ (for S) and onto the space C^{n-1} of the variables $z_2, \cdots, z_n$ (for σ and D). Evidently $\sigma_{z_1^0}$, $S_{z_1^0}$, σ and S are closed sets. By Ω we denote the set of points of the z_1-plane for which $\sigma_{z_1} \setminus D_{z_1} \neq \emptyset$. Ω is a closed set, since σ is closed and D is open. Finally we denote by Ω' the set complementary to Ω with respect to the entire plane C_{z_1}. It is evident that Ω' is an open set and that all (finite) points $z_1 \notin s$ belong to the set Ω'.

On the set Ω' we define functions $R_j(z_1)$ $(j = 1, \cdots, N)$ by the following equations:

$$R_j(z_1^0) = \begin{cases} 0, & \text{if} \quad z_1^0 \notin s, \\ \sup |w_j - f_j(z)| \text{ for } (z, w) \in S \cap \{z_1 = z_1^0\}, & \\ & \text{if} \quad z_1^0 \in s. \end{cases} \tag{1.22}$$

LEMMA 6. *$R_j(z_1)$ is a logarithmically subharmonic function.* [1])

PROOF. We will prove our assertion for the function $R_1(z_1)$. First we show that the function $\ln R_1(z_1)$ is upper semicontinuous. It is sufficient to establish this for the function $R_1(z_1)$ itself at points $z_1 \in s$. Let points $z_1^{(p)}$ be in s

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1) For the definition of logarithmically subharmonic functions, see I. I. Privalov, *Subharmonic functions*, ONTI, Moscow, 1937, p. 60 (Russian).

where, for $p = 1, 2, \cdots$, $\lim_{p\to\infty} z_1^{(p)} = z_1^0 \in s$ and $\overline{\lim}_{p\to\infty} R_1(z_1^{(p)}) = \alpha$, with α a constant. We must show that $\alpha \leq R_1(z_1^0)$.

To each value $z_1 = z_1^{(p)}$ there should correspond at least one point $M_p(z_1^{(p)}, \cdots, z_n^{(p)}, w_1^{(p)}, \cdots, w_N^{(p)}) \in S$ such that

$$R_1(z_1^{(p)}) = |w_1^{(p)} - f_1(z_1^{(p)}, \ldots, z_n^{(p)})|. \tag{1.23}$$

We consider the point $M(z_1^0, \zeta_2, \cdots, \zeta_n, \omega_1, \cdots, \omega_N) = \lim_{p\to\infty} M_p$. Since S is a closed set, M is in S. Therefore, in view of (1.22) and (1.23), we have

$$\alpha = \overline{\lim_{p\to\infty}} |w_1^{(p)} - f_1(z_1^{(p)}, \ldots, z_n^{(p)})| \\ = |\omega_1 - f_1(z_1^0, \zeta_2, \ldots, \zeta_n)| \leqslant R_1(z_1^0).$$

Thus the upper semicontinuity of the function $R_1(z_1)$ is proved.

We consider a disk $\gamma = \{|z_1 - z_1^0| < \rho\} \subset \Omega'$ with its center at a point $z_1^0 \in s$. Our lemma will be proved if we show that the following inequality is impossible:

$$\frac{1}{2\pi}\int_0^{2\pi} \ln R(z_1^0 + \rho e^{i\varphi})\, d\varphi < \ln R(z_1^0)$$

(where $z_1 - z_1^0 = \rho e^{i\theta}$, and the integral is taken in the sense of Lebesgue).

Suppose that this inequality holds. By the use of Poisson's formula, we construct in the disk γ a holomorphic function $\psi(z_1)$ (here $|\phi(z_1)|$ will be continuous and differ from zero in the closed disk $\overline{\gamma}$) satisfying the following condition on the circle $\partial\gamma$:

$$\ln|\psi(z_1)| = -\ln R_1(z_1) - \ln R_1(z_1^0) + \frac{1}{2\pi}\int_0^{2\pi} R_1(z_1^0 + \rho e^{i\varphi})\, d\varphi.$$

Then $R(z_1^0)\,|\psi(z_1^0)| = 1$ while, by assumption, on the circle we have

$$R_1(z_1)\,|\psi(z_1)| < 1.$$

By means of the function $\psi(z_1)$ we construct a family of analytic surfaces

$$(\chi_t) \qquad [w_1 - f_1(z)]\,\psi(z_1) = e^{i\theta}(1 + t). \tag{1.24}$$

Here the point z in question is in D and it is assumed that $0 \leq t < \infty$; θ is a constant and below we shall indicate how to choose it.

We show that this family would then have the three properties enumerated in Lemma 5; as we have seen, this is impossible, and leads us to a contradiction

which therefore refutes our assumption.

1) The boundary of the surface χ_t may consist of points of two types: the condition $z \in \partial D$ must be satisfied for one, and $z_1 \in \partial\gamma$ for the other (i.e., we must have $|z_1 - z_1^0| = \rho$). The former is necessarily projected onto the disk $\{|z_1 - z_1^0| < \rho\} \Subset \Omega'$.

By the definition of the set Ω', points of the set σ which are projected onto Ω' are in the domain D. If S is a sufficiently small neighborhood of the set $\widetilde{S}$, its projection onto the space C_z^n and the boundary ∂D have no common points. Hence it follows that in the neighborhood of S there are no boundary points of the surfaces χ_t of the first type.

Turning to the boundary points of the second type, we first of all remark that for all points $(\zeta_1, \cdots, \zeta_n, \omega_1, \cdots, \omega_N) \in S$ we have, in view of (1.23), that for $|\zeta_1 - z_1^0| = \rho$,

$$|\omega_1 - f_1(\zeta_1, \cdots, \zeta_n)|\,\psi(\zeta_1) < 1.$$

Hence it follows from equation (1.24) that the boundary points of the surface χ_t of the second type also lie outside the neighborhood $\widetilde{S}$.

2) For all values of the parameter t in the semi-interval $0 \leq t < \infty$, the surfaces χ_t lie outside the set $\overline{\Sigma}$. In fact, if a point $(z, w) \in \overline{\Sigma}$, then $w_1 = f_1(z)$, while on the surfaces χ_t equation (1.24) is satisfied.

3) The point z_1^0 is in s. Therefore there exists a point

$$M_0(z_1^0, \cdots, z_n^0, w_1^0, \cdots, w_N^0) \in S$$

such that $R_1(z_1^0) = |w_1^0 - f_1(z_1^0, \cdots, z_n^0)|$. Then $|w_1^0 - f_1(z_1^0, \cdots, z_n^0)|\psi(z_1^0) = 1$ and, because of equation (1.24), for a suitable choice of the constant θ the surface χ_0 passes through the point M_0. On the other hand, it may be always assumed that the function $f(z)$ is bounded in the domain D. In addition $|w_1| < r_1$ for all points of the set S. Hence it follows that for sufficiently large values of the parameter t the surface χ_t has no points in common with the set S.

Therefore, the family of surfaces $\{\chi_t\}$ must possess all the three properties stated in Lemma 5. This is impossible and thus Lemma 6 has been proved.

COROLLARY 1. *Let the set Ω' be decomposed into domains $\Omega_1', \Omega_2', \cdots$. In each of these domains the functions $R_j(z_1)$ are logarithmically subharmonic. Suppose that one of these domains, say Ω_1', contains a point $z_1^0 \notin \overline{s}$. Then all all the functions $R_j(z_1)$ become zero in some neighborhood of this point. Since the $R_j(z_1)$ are logarithmically subharmonic, they are, in this case, identically*

equal to zero in the domain Ω_1'.

COROLLARY 2. *If* $R_j(z_1^0) = 0$, *where* $z_1^0 \in \Omega' \cap s$, *then* $\Sigma_{z_1^0} = S_{z_1^0}$ *(for a point* $z_1^0 \notin s$ *both sets are empty). Here* $\Sigma_{z_1^0}$ *(analogously* $S_{z_1^0}$*) is the projection onto the space* C^{n+N-1} *of the variables* $z_2, \cdots, z_n$, w *of the section of the set* Σ *by the plane* $z_1 = z_1^0$.

In fact, the point z_1^0 is in Ω'; accordingly, the projections of all the points of the section $S \cap \{z_1 = z_1^0\}$ onto the space C_z^n belong to the domain D. If the point $(z_1^0, \cdots, z_n^0, w_1^0, \cdots, w_N^0) \in S$, while the point $(z_1^0, \cdots, z_n^0) \in \overline{\Delta} \subset D$, then from the conditions $R_j(z_1^0) = 0$ it follows that in our case $w_j^0 = f_j(z_1^0, \cdots, z_n^0)$, which simply expresses our assertion. If the point $(z_1^0, \cdots, z_n^0) \in D - \overline{\Delta}$, then for at least one of the functions f_j, say for the function f_1, it turns out that $|f_1(z_1^0, \cdots, z_n^0)| > 1$. Then, in view of the condition $R_1(z_1) = 0$, we have

$$|w_1^0| = |f_1(z_1^0, \cdots, z_n^0)| > 1.$$

This means that the point of the set S in question does not belong to the closed polycylinder $\{|z_i| \le z_i, |w_j| \le 1;\ i = 1, \cdots, n;\ j = 1, \cdots, N\}$, which is however impossible. Hence our assertion again follows.

Before continuing the investigation we will first cite another proposition from the theory of subharmonic functions.

LEMMA 7. *If* $\phi(z)$ *is a subharmonic function in a domain* B *of the* z*-plane, then on the approach of the point* $z \in B$ *to a point* $\zeta \in B$ *along a line* $L \subset \overline{B}$ *we have*

$$\overline{\lim_{z \to \zeta}}\ \phi(z) = \phi(\zeta);$$

here it is assumed that the line L *has no multiple points.*

PROOF. Let $\zeta = 0$. We construct disks $\gamma_1, \gamma_2 \subset B$ with their centers at the point ζ in such a way that $\gamma_2 \subset \gamma_1$. Without loss of generality we may assume that γ_1 is the unit disk. Then the disk γ_2 has a radius $\rho < 1$. We denote by L_0 the portion of the line L between the points where it meets the circle $\partial\gamma_1$ for the first time (on the way from z to ζ) and the circle $\partial\gamma_2$ for the last time. The line L_0, generally speaking, decomposes the disk γ_1 into several domains; their boundaries consist of portions of the circles $\partial\gamma_1$ and the line L_0. We denote by A the part of them which contains the point ζ. Let $l = L_0 \cap \partial A$. Since the line L has no multiple points, A is a simply-connected domain.

By means of a function $z = \Phi(w)$ we conformally map the domain A onto the

disk $|w| < 1$. Put $\Phi(0) = 0$; then for $|w| < 1$ the function $\Phi(w)/w$ is holomorphic and distinct from zero (it takes on the value $\Phi'(0)$ at $w = 0$). We consider the quantity

$$\ln|\Phi'(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|\Phi(e^{i\mu})|d\mu. \tag{1.25}$$

To evaluate the integral (1.25) we need the limiting values of the quantity $|\Phi(w)|$ on the circle $|w| = 1$. Evidently they exist, and are continuous and different from zero.

We denote by $2\pi a$ the measure of those portions of the circle $\{|w| = 1\}$ on which $|\Phi(w)| < 1$; ρ (the radius of the disk γ_2) is the minimum value of this modulus on the above-mentioned portions of the circle. From equation (1.25) it results that $\rho < |\Phi'(0)|$. On the other hand, by Koebe's theorem,[1] $\rho > |\Phi'(0)|/4$. It follows from the last two inequalities that $\rho^{1-a} \geq 1/4$, and therefore that $a \longrightarrow 1$ as $\rho \longrightarrow 0$.

Since the function $\phi(z)$ is upper semicontinuous for $|z| \leq 1$, there exist numbers M and N such that 1) $\phi(z) \leq M$ for $|z| \leq 1$ and 2) $\phi(z) \leq N$ for $|z| = 1$. Considering $\int_0^{2\pi}\phi(\Phi(e^{i\mu}))d\mu$ we find that

$$aN + (1 - a)M \leq \phi(0).$$

Hence as $\rho \longrightarrow 0$ (and so $a \longrightarrow 1$) we obtain $N \leq \phi(0)$ and consequently $\overline{\lim}_{z\to\zeta}\phi(z) \leq \phi(\zeta)$. But the function $\phi(z)$ is upper semicontinuous at the point ζ; therefore the inequality sign can be discarded and consequently the equality $\overline{\lim}_{z\to\zeta}\phi(z) = \phi(\zeta)$ is valid. Lemma 7 is thus proved.

6. Sets consisting of Weil analytic polyhedra. Now we can prove the proposition forming the basis of our final deductions.

LEMMA 8. $\pi_1(\Sigma) = \Sigma$.

More precisely, we consider the closed set $\overline{\Delta} = \{|f_j(z)| \leq 1,\ j = 1, \cdots, N\} \Subset D \subset \{|z_i| < r_i,\ i = 1, \cdots, n\} \subset C^n$. Suppose that all the functions $f_j(z)$ are holomorphic in the domain D. In the space $C^{n+N}_{z,w}$ of the variables $z_1, \cdots, z_n$, $w_1, \cdots, w_N$ we construct the closed set $\Sigma = \{w_j = f_j(z),\ z \in \overline{\Delta},\ j = 1, \cdots, N\} \subset \overline{C} = \{|z_i| \leq r_i,\ |w_j| \leq 1;\ i = 1, \cdots, n;\ j = 1, \cdots, N\}$.

We assert that $\pi_1(\Sigma) = \Sigma$, i.e., that *the set Σ can be approximated from the outside by a sequence of Weil sets* $F_\nu(\nu = 0, 1, 2, \cdots, F_\nu \supset F_{\nu+1})$. The Weil

[1] See, for example, I. I. Privalov, *Introduction to the theory of functions of a complex variable*, Gostehizdat, Moscow, 1954, p. 420 (Russian).

sets are defined in the same way as the set Δ; the functions $f_j(z)$ are to be replaced by polynomials.

PROOF. We first suppose that the lemma is true for the space C^{n-1} and show that then it is also true for the space C^n. Next we consider the case $n = 1$. The set $\pi_1(\Sigma) = S$ and hence the set s are bounded. If the set Ω' coincides with the whole z_1-plane, then the assertion of our lemma for the space C^n is evident from the Corollaries 1 and 2 of Lemma 6.

Suppose that the assertion of Lemma 8 for the space C^n is not true and that $\Omega' \neq C'_{z_1}$. Then: 1) Each of the domains Ω'_k (into which the set is decomposed) has finite boundary points; 2) among these domains there exists at least one (say the domain Ω'_1) containing the point $z_1 \notin s$. Let a (finite) point ζ be in $\partial\Omega'_1$ (thus the point $\zeta_1 \in \Omega$; we choose it in such a way that it is an exterior point for the domains Ω'_k with $k \neq 1$).

Then the set $\Sigma_\zeta \neq \emptyset$. In actual fact there may occur two cases: 1) $\zeta \in (s - \partial s)$. Then $\zeta = \lim_{\nu\to\infty}\zeta_\nu$, where $\zeta_\nu \in (\Omega'_1 \cap s)$, $\nu = 1, 2, \cdots$, and $\Sigma_{\zeta_\nu} = S_{\zeta_\nu}$. But $S_{\zeta_\nu} \neq \emptyset$, namely, $\Sigma_{\zeta_\nu} \neq \emptyset$. Hence, as Σ is a closed set, it follows that $\Sigma_{\zeta_\nu} \neq \emptyset$.

2) $\zeta \in \partial s$. In this case, if $\Sigma_\zeta = \emptyset$, then $\Sigma_{z_1} = \emptyset$ for $z_1 \in U_\zeta$, since Σ is a closed set. Here U_ζ is some neighborhood of the point ζ on the z_1-plane. In this neighborhood there exist both interior and exterior points for the set s. Let a point $\zeta_0 \notin s$ lie in this neighborhood, as well as in a disk $|z_1 - \zeta_0| < \delta$ which already contains points of the set s. Then the function $\delta(z_1 - \zeta_0)^{-1}$ is holomorphic on the set S, namely, it is holomorphic also in some of its $(n+N)$-complex-dimensional neighborhoods $\tilde{S} \in \pi_0$. There the function $\delta(z_1 - \zeta_0)^{-1}$ can be replaced by a polynomial Φ with any required degree of accuracy. We choose this polynomial in such a way that $|\Phi| < 1$ for $(z, w) \in \Sigma$ (since $\delta|z_1 - \zeta_0|^{-1} < 1$ on the set Σ), while in the set S there exist points for which $|\Phi| > 1$. This result contradicts the definition of the set $S = \pi_1(\Sigma)$; the set Σ_ζ in the case at hand cannot be empty.

Since $\zeta \in \Omega$, the set S_ζ contains at least one point whose projection onto the space C^n_z does not fall in the domain D. Every point of the set Σ is projected into the domain D; therefore in the domain S_ζ we can find an interior point M which is exterior to the set Σ_ζ. It is easy to see that the set $\Sigma_\zeta \neq \emptyset$ in the space C^{n+N-1} of the variables $z_2, \cdots, z_n$, w is defined in the same way as the set Σ in its space. Then, in view of our assumption, $\pi_1(\Sigma_\zeta) = \Sigma_\zeta$ and by the definition of the class π_1 there exists the set

$$\Delta_\tau = \{|P_i(z_2, \ldots, z_n, w)| < \tau, \quad i = 1, \ldots, m\} \subset \Sigma_\zeta,$$

which, however, does not contain the point M. Here τ is a parameter and we shall consider the sets Δ_α and Δ_β corresponding to the values of the parameter $\tau = \alpha$, β with $\alpha > \beta$. Next we define the sets

$$G_\tau = \{|z_1 - \zeta| < \delta \qquad (z_2, \ldots, z_n, w) \in \Delta_\tau\},$$
$$H = \{|z_1 - \zeta| < \delta \qquad (z_2, \ldots, z_n, w) \in \Sigma_{z_1}\}.$$

Here the number δ may be chosen so small that: 1) the projection of the set G_α onto the space C_z^n lies inside the domain D; 2) $H \subset G_\beta$; 3) the disk $\{|z_1 - \zeta| < \delta\} \subset \Omega \cup \Omega_1'$.

In the disk $|z_1 - \zeta| < \delta$ we now define a function $\lambda(z_1)$. Put

$$\lambda(z_1) = \begin{cases} \max[\beta, \max|P_i(S_{z_1})| i = 1, \ldots, m] & \text{for} \quad z_1 \in s, \\ \beta \quad \text{for} \quad z_1 \notin s. \end{cases}$$

We will show that $\lambda(z_1)$ is a logarithmically subharmonic function.

The upper semicontinuity of the function $\lambda(z_1)$ on the set s follows from that of $\max|P_i(S_{z_1})|$ on the same set. The latter is established in exactly the same way as for the upper semicontinuity of the function $R_j(z_1)$ in Lemma 6.

Suppose that the function $\lambda(z_1)$ is not logarithmically subharmonic. We repeat the argument which, in Lemma 6, was used to prove the analogous property of the function $R_j(z_1)$.

Suppose also that the closed disk $|z_1 - z_1^0| \le \rho$ lies entirely outside the disk $|z_1 - \zeta| < \delta$ and that

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\lambda(z_1^0 + \rho e^{i\mu})\, d\mu < \ln\lambda(z_1^0).$$

We construct (as in the proof of Lemma 6) a function $\psi(z_1)$ that is holomorphic and different from zero in the closed disk $|z_1 - z_1^0| \le \rho$. As such a function we choose

$$\lambda(z_1)|\psi(z_1)| \begin{cases} < 1 & \text{for} \quad |z_1 - z_1^0| = \rho, \\ = 1 & \text{for} \quad z_1 = z_1^0. \end{cases}$$

Since it is always true that $\lambda(z_1) > \beta$, we have $\beta|\psi(z_1)| < 1$ for $|z_1 - z_1^0| = \rho$, and so also for $z_1 = z_1^0$. Then $\lambda(z_1^0) > \beta$. Hence it follows that

$\lambda(z_1^0) = \max\left|P_i\left(S_{z_1^0}\right)\right|$, where $i = 1, \cdots, m$. To be more definite, let us assume that this maximum is attained for $i = 1$; then $\lambda(z_1^0) = \max\left|P_1\left(S_{z_1^0}\right)\right|$.

We consider the family of analytic surfaces

$$(\chi_t) \qquad P_1(z, w)\psi(z_1) = e^{i\theta}(1 + t). \tag{1.26}$$

Here $0 \leq t < \infty$, $|z_1 - z_1^0| < \rho$, and θ is a constant whose choice will be indicated below. We will show that the family $\{\chi_t\}$ must have the three properties mentioned in Lemma 5.

1) The family $\{\chi_t\}$ has no boundary points lying in the neighborhood $\widetilde{S}$. In fact, the boundary points of surfaces χ_t are projected onto the circle $|z_1 - z_1^0| = \rho$. But there $\lambda(z_1)\,|\psi(z_1)| < 1$. Therefore for points $(z, w) \in \widetilde{S}$, which are projected onto this circle, $|P_1(z, w)\psi(z_1)| < 1$. This is inconsistent with equation (1.26).

2) The surfaces χ_t cannot possess points in common with the set $\overline{\Sigma}$. In the closed disk $|z_1 - z_1^0| \leq \rho$ we have

$$\max|P_1(\Sigma_{z_1})| < \beta$$

(since $H \subset G_\beta$) and $|\psi(z_1)| < 1/\beta$. Therefore $|P_1(z, w)\psi(z_1)| < 1$ for $(z, w) \in \overline{\Sigma}$, which is inconsistent with equation (1.26).

3) For a suitable choice of the constant θ the surface χ_0 necessarily has points in common with the set S, since $\max\left|P_1\left(S_{z_1^0}\right)\right|\psi(z_1^0) = 1$.

Thus the family $\{\chi_t\}$ must have the three properties of Lemma 5. This is impossible and consequently, contrary to assumption, the function $\lambda(z_1)$ is logarithmically subharmonic.

In the disk $|z_1 - \zeta| < \delta$ we consider a set of points $A = \{\lambda(z) \geq \alpha\}$. This set is closed, since the function $\lambda(z)$ is upper semicontinuous. Let a point $a \in (\Omega_1' \cap \{|z_1 - \zeta| < \delta\})$. Then $\lambda(a) = \beta$ (if the point $a \notin s$, this follows immediately from the definition of the function $\lambda(z_1)$; if the point $a \in s$, then $S_a = \Sigma_a$ and the set $H \subset \Delta_\beta$, and hence $\max|P_1(S_a)| < \beta$ when the equality again holds). Therefore the point $a \notin A$. We consider a non-Euclidean disk γ with its center at the point a (relative to the absolute disk $|z_1 - \zeta| \leq \delta$), not containing points of the set A, but having them on its circumference $\partial\gamma$. We observe that the point $\zeta \in A$ and that the set A is closed in the closed disk $|z_1 - \zeta| \leq \delta$. If the point $z_1 \in \gamma$, then $\lambda(z_1) < \alpha$; in this case the set $S_{z_1} \subset \Delta_\alpha$ and accordingly it is projected into the domain D_{z_1} of the space C^{n-1} of the variables $z_2, \cdots, z_n$. This means that $z_1 \in \Omega_1'$ and $\Sigma_{z_1} = S_{z_1}$. Thus $S_{z_1} \subset \Delta_\beta$ and $\lambda(z_1) = \beta$. Here

z_1 is an arbitrary point of the disk γ. By Lemma 7 this equality must also hold at all points of the circle $\partial\gamma$.

This last conclusion necessarily contradicts the fact that $\partial\gamma \cap A \neq \emptyset$, and the contradiction forces us to reject the assumption we have made. Thus we have proved that $\Sigma = S$.

It remains to consider the case $n = 1$. Then D, Δ and $\sigma = s$ are sets on the z-plane. We define functions $R_j(z)$ in exactly the same way as in Lemma 6. It is easy to see that in this case there is no point $z \in \Omega$, since S is the smallest set of the class π_1 containing the set Σ. Hence it follows that $\Omega = \emptyset$, $\Omega' = C'_z$ and therefore that $R'_j(z) \equiv 0$. Then, in view of the corollary of Lemma 6, we have $\Sigma = S$. With this, Lemma 8 is completely proved.

LEMMA 9. *Let Weil's bounded analytic polyhedron Δ be one of the connected components of the set $\Delta^* = \{|f_j(z)| < 1,\ j = 1, \dots, N\} \Subset D \subset C^n$. Here the $f_j(z)$ are functions holomorphic in a domain D of the space C^n of the variables $z_1, \dots, z_n$.*

Then every function $F(z)$ holomorphic in the closed polyhedron $\bar{\Delta}$ can be represented inside this polyhedron by means of the uniformly convergent series

$$F(z) = \sum_{\nu=0}^{\infty} P_\nu(z_1, \dots, z_n, f_1(z), \dots, f_N(z)). \tag{1.27}$$

Here $P_\nu(z, f)$ are polynomials in $z_1, \dots, z_n, f_1(z), \dots, f_N(z)$.

PROOF. In the space $C^{n+N}_{z,w}$ of the variables $z_1, \dots, z_n, w_1, \dots, w_N$ consider the set

$$\Sigma^* = \{z \in \Delta^*,\ w_j = f_j(z),\ j = 1, \dots, N\}.$$

By the preceding lemma, $\pi_1(\Sigma^*) = \Sigma^*$. Therefore there exists a neighborhood $\tilde{\Sigma}^* \in \pi_0$ such that in its connected components $\tilde{\Sigma} \supset \Delta$ the function $F(z)$ remains holomorphic. By Theorem 22.4, (I) the function $F(z)$ can be represented in the domain $\tilde{\Sigma}$ by a uniformly convergent series of polynomials

$$F(z) = \sum_{\nu=0}^{\infty} P_\nu(z_1, \dots, z_n, w_1, \dots, w_N).$$

Setting $w_j = f_j(z)$, we obtain at points $z \in \Delta$ the representation (1.27).

REMARK. If the functions $f_j(z)$ belong to some complete family of functions Φ, then, as is seen immediately from the definition of the complete family, each

term of the series (1.27) also belongs to the same family.

7. Completion of the proof of Theorem 2.1. We shall show the sufficiency of the condition, stated in Theorem 2.1, for an arbitrary complete family Φ of holomorphic functions. For this purpose we repeat verbatim the argument which was carried out in subsection 2 of the present section for the case of polynomials. The only difference consists of replacing the polynomials $\phi_j^{(p)}(z)$ by the functions $f_j^{(p)}(z) \in \Phi$ and the series of polynomials $\Sigma_{k=0}^{\infty} P_k(z)$ by the series (1.27) in Lemma 9.

Thus Theorem 2.1 has been completely proved.

REMARK. As already noticed, the proof of Theorem 2.1 in the case of bounded domains is carried out without using the chordal metric. The use of this metric in the case of unbounded domains considerably simplifies the proof.

8. Cases of domains over the space C^n. Let D be a finite-sheeted domain over the space C_z^n. We consider an open set of points $\{|z_i| < r_i, |f_j(z)| < 1;$ $i = 1, \cdots, n;\ j = 1, \cdots, N\} \Subset D$ where the $f_j(z)$ are functions holomorphic in the domain D. Let a domain Δ be one of the connected components of this set. We assume that all the functions $f_j(z)$ are essential to specify the domain Δ. Its boundary $\partial\Delta$ consists of hypersurfaces $\sigma_j = \partial\Delta \cap \{|f_j(z)| = 1\}$. If the intersections $\sigma_{j_1} \cap \sigma_{j_2}$ are not hypersurfaces, the domain Δ (by the analogy with the case of the space C^n) is called a Weil analytic polyhedron over the space C^n. We have

LEMMA 10. *Let the Weil analytic polyhedron Δ over the space C_z^n be one of the connected components of the set $\{|z_i| < r_i, |f_j(z)| < 1, i = 1, \cdots, n; j = 1, \cdots, N\} \Subset D$. Here D is a finite-sheeted domain over the space C^n which is strongly convex relative to some complete family Φ of holomorphic functions; all functions $f_j(z) \in \Phi$.*

Then every function $F(z)$ holomorphic in the closed polyhedron $\overline{\Delta}$ can be represented inside this polyhedron by a uniformly convergent series which consists of functions belonging to the family Φ.

PROOF. In view of the condition of strong Φ-convexity of the domain D, for every pair of points $z', z'' \in D$ with identical coordinates there exists a function $\phi(z) \in \Phi$ having distinct functional elements at these points. Then, by Borel's Lemma, there exist functions $\phi_k(z) \in \Phi$, $k = 1, \cdots, \mathfrak{N}$, and a number $\epsilon > 0$ such that for every pair of points $z', z'' \in \overline{\Delta}$ with identical coordinates one can find at least one function $\phi_K(z)$ satisfying the condition

$$|\phi_K(z') - \phi_K(z'')| \geq \epsilon. \tag{1.28}$$

In the space $C^{n+\mathfrak{N}}_{z,\zeta}$ of the variables $z_1, \cdots, z_n, \zeta_1, \cdots, \zeta_{\mathfrak{N}}$ we consider the analytic surface

$$\mathfrak{F} = \{z \in \bar{\Delta},\ \zeta_k = \varphi_k(z),\quad k = 1, \ldots, \mathfrak{N}\}.$$

We easily see that an arbitrary pair of distinct points of the closed polyhedron $\bar{\Delta}$ (including one with identical coordinates) corresponds to distinct points of the surface $\mathfrak{F}$. Moreover, evidently $\mathfrak{F} \subset \bar{\mathfrak{D}}$, where

$$\mathfrak{D} = \left\{z \in \Delta,\ |\zeta_k - \varphi_k(z)| < \frac{\varepsilon}{3},\quad k = 1, \ldots, \mathfrak{N}\right\}$$

is a bounded analytic polyhedron in the space $C^{n+\mathfrak{N}}_{z,\zeta}$.

Every holomorphic function $F(z)$ given on the closed polyhedron $\bar{\Delta}$ is defined in terms of the equations $\zeta^0_k = \phi_k(z)$, $k = 1, \cdots, \mathfrak{N}$, and a function $\tilde{F}(z_1, \cdots, z_n, \zeta^0_1, \cdots, \zeta^0_{\mathfrak{N}}) = F(z)$ at points $(z_1, \cdots, z_n, \zeta^0_1, \cdots, \zeta^0_{\mathfrak{N}}) \in \mathfrak{F}$. We can continue this function to the entire closed polyhedron $\mathfrak{D}$ by setting, for the point $(z_1, \cdots, z_n, \zeta_1, \cdots, \zeta_{\mathfrak{N}}) \in \mathfrak{D}$,

$$F(z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_{\mathfrak{N}}) = \tilde{F}(z_1, \ldots, z_n, \zeta^0_1, \ldots, \zeta^0_{\mathfrak{N}}) = F(z), \tag{1.29}$$

where the point $(z_1, \cdots, z_n, \zeta^0_1, \cdots, \zeta^0_{\mathfrak{N}}) \in \mathfrak{F}$. In general, on the surface $\mathfrak{F}$ there are several points with their first n coordinates equal to $z_1, \cdots, z_n$. In equation (1.29) we select from these points those for which the coordinates $\zeta^0_1, \cdots, \zeta^0_n$ satisfy the condition: $|\zeta_k - \zeta^0_k| \leq \epsilon/3$, $k = 1, \cdots, \mathfrak{N}$. Because of the inequalities (1.28), this requirement uniquely defines the point $(z, \zeta^0) \in \mathfrak{F}$ corresponding to the point $(z, \zeta) \in \bar{\mathfrak{D}}$. Evidently $F(z_1, \cdots, z_n, \zeta_1, \cdots, \zeta_{\mathfrak{N}})$ is a holomorphic function in the closed polyhedron $\bar{\mathfrak{D}}$.

By applying Lemma 9, we represent the function $\tilde{F}$ inside the polyhedron $\mathfrak{D}$ by a uniformly convergent series

$$\begin{aligned}\tilde{F}&(z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_{\mathfrak{N}})\\ &= \sum_{\nu=0}^{\infty} P_\nu(z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_{\mathfrak{N}}, f_1, \ldots, f_N,\\ &\qquad \zeta_1 - \varphi_1(z), \ldots, \zeta_n - \varphi_n(z)).\end{aligned} \tag{1.30}$$

Here each of the functions P_ν is a polynomial in all its arguments. Setting $\zeta_k = \phi_k(z)$, $k = 1, \cdots, \mathfrak{N}$, in equation (1.30) we obtain a series converging uniformly in the polyhedron Δ:

$$F(z)=\sum_{\nu=0}^{\infty} P_\nu(z_1, \ldots, z_n, \varphi_1(z), \ldots, \varphi_{\mathfrak{N}}(z), \\ f_1(z), \ldots, f_{N'}(z), 0, \ldots, 0). \tag{1.31}$$

This series satisfies all the conditions formulated in our assertion. Lemma 10 is proved.

Using Lemma 10 instead of Lemma 9 and repeating with the argument in subsections 2 and 7 of the present section, we can prove

THEOREM 2.2 (Behnke-Stein [1], (I)). *Let D be a domain over the space C^n having a finite-sheeted holomorphy hull $H(D)$, and let Φ be a complete family of functions holomorphic in the domain D. An arbitrary function $F(z)$ holomorphic in the domain D can be represented as the sum of a series converging uniformly in D and consisting of functions which belong to the family Φ, if and only if the holomorphy hull $H(D)$ of the domain D is strongly convex relative to the family Φ.*

§3. RUNGE DOMAINS AND THEIR GENERALIZATIONS

1. Runge domains relative to a complete family of holomorphic functions. Runge's theorem in the theory of functions of one complex variable states: *Let $\mathfrak{O}_D$ be the ring of functions holomorphic in a domain $D \subset C^1$. Then every function $f \in \mathfrak{O}_D$ can be uniformly approximated in the domain D by a polynomial if and only if the domain D is simply-connected.*

For $n > 1$ a similar assertion is proved to be false (see subsection 2 of the present section). The following question may arise: when can the function $f \in \mathfrak{O}_D$ be uniformly approximated in the domain $D \subset C^n$ by polynomials?

This problem in turn forms a part of a more general problem considered in the preceding section: when can the function $f \in \mathfrak{O}_D$ be uniformly approximated in the domain D over the space C^n by functions belonging to some complete family Φ of functions holomorphic in that domain D?

An answer to the latter question is given by Theorem 2.1 and 2.2. However, the conditions indicated by these theorems are not of a purely geometric character, in essential distinction to Runge's theorem, and this makes their application difficult. In several cases one can obtain more convenient criteria which will be indicated below in the present section and in §16 of Chapter III.

DEFINITION (Runge domain relative to a complete family of functions). Let D be a domain over the space C^n and Φ a complete family of functions holomorphic in this domain. If any function $f \in \mathfrak{O}_D$ can be represented as the limit of a

sequence, converging uniformly in D, of functions in the family Φ, then D is called a *Runge domain relative to the family* Φ.

Theorem 2.2 can be expressed in the following way in this new terminology:

A bounded and finite-sheeted domain D over the space C^n is a Runge domain relative to some complete family Φ of holomorphic functions if and only if its holomorphy hull $H(D)$ is strongly convex relative to this family.

We shall note some special cases. Runge domains relative to the family of all polynomials and of all rational functions are called, respectively, *Runge domains of the first and second kinds.* If D and D^* are domains of holomorphy over the space C^n, while $D < D^*$, and D is a Runge domain relative to the family of functions of $\mathfrak{D}_{D^*}$, then we say briefly that D is a *Runge domain relative to the domain* D^* and we write $D\mathfrak{R}D^*$.

From this definition it immediately follows that:

If $D\mathfrak{R}D_1$ *and* $D_2\mathfrak{R}D^*$, *while* $D < D_2 < D_1$, *then* $D\mathfrak{R}D^*$.

The definition of the relation $D\mathfrak{R}D^*$ can be extended without any change to the case when D and D^* are not assumed to be domains of holomorphy. However such a generalization merely leads to complicated statements of the theorems. Instead of the domains D and D^* we speak of their hulls $H(D)$ and $H(D^*)$.

2. Runge domains of the first and second kinds. A necessary and sufficient condition for the domain $D \subset C^n$ to be a Runge domain of the first kind is indicated by Theorem 2.1. The Runge domain of the first kind is necessarily finite, since the domain of holomorphy of any polynomial is the space C^n. The holomorphy hull $H(D)$ of this domain D coincides with its K-convex hull $K(D)$; here K is the complete family (class) of polynomials. Hence it follows that the domain $H(D)$ is finite and single-sheeted (as a subdomain of the space C^n). Thus the following theorem holds:

THEOREM 3.1. *If the domain has an infinite or multiple-sheeted holomorphy hull, then it is not a Runge domain of the first kind.*

The domain of holomorphy of any rational function is a certain subdomain of the space P^n. The holomorphy hull $H(D)$ of the Runge domain of the second kind D coincides with its K-convex hull. Here K is the complete family of rational functions. Hence it follows that the hull $H(D)$ is single-sheeted. Thus the following theorem holds:

THEOREM 3.2. *If the domain D has a many-sheeted holomorphy hull, then it*

is not a Runge domain of the second kind.

Theorems 3.1 and 3.2 make it possible to construct examples of those (including simply-connected) domains of the space C^n for $n > 1$ which are not Runge domains of either the first or the second kind. To this end it is sufficient that such domains should have many-sheeted holomorphy hulls. In §13.5, Chapter II, (I) we gave an example of such domains of the space C^2 (the semidisk domain).

We now consider some examples.

1) *The polycylindrical domain* $D = D_1 \times D_2 \times \cdots \times D_n$, *where each of domains* $D_k \subset C^1_{z_k}$ *is simply-connected, is a Runge domain of the first kind.* In fact, let $w = f(z_k)$ be a function which maps the domain D_k conformally onto the disk $|w| < 1$. Then for any domain $\widetilde{D} \Subset D$ there exists a domain $D^* = \{|f(z_k)| < \eta_k, k = 1, \cdots, n, 0 < \eta_k < 1\}$ such that $\widetilde{D} \Subset D^* \Subset D$. Applying Runge's theorem for functions of a single variable, we may replace each of the functions $f(z_k)$ by a polynomial $\phi(z_k)$ with any degree of accuracy. As a result we obtain a domain $D^{**} = \{|\phi(z_k)| < \mu_k, \mu_k > 0, k = 1, \cdots, n\}$ such that $\widetilde{D} \Subset D^{**} \Subset D$. Applying A. Weil's Theorem 22.3, (I) to the domain D^{**}, we can prove our assertion.

2) We have observed above that for $n = 1$ the Runge domain of the first kind is always simply-connected, while for $n > 1$ the union of Runge domains of the first kind is not identical with that of finite simply-connected domains. In this connection it should be noted that the polyhedron $D = \{|wz - 1| < 1/2, |w^2 z| \le 2, |z| < 1\} \subset C^2_{w,z}$ is a Runge domain of the first kind by virtue of A. Weil's Theorem 22.4, (I). However, the closed line $\{w = 4e^{i\theta}/3, z = 3e^{-i\theta}/4, 0 \le \theta \le 2\pi\} \subset D$ does not shrink to a point within the limits of this domain. Thus the domain D is not simply-connected.

3) We now construct, following K. Oka [2], (I), an example of a domain of holomorphy which is not even a Runge domain of the second kind. In the space $C^2_{w,z}$ consider the polycylindrical domain $\Gamma = \Gamma_1 \times \Gamma_2$, where $\Gamma_1 = \{r_1 < |w| < 1\}$, $\Gamma_2 = \{r_2 < |z| < 1\}$ and the analytic plane $E = \{w - z - 1 = 0\}$. Assume that $r_1 + r_2 > 1$. Then the intersection $\Gamma \cap E$ is decomposed into two disconnected parts. In fact, if a point $(w_0, z_0) \in \Gamma \cap E$, then the point w_0 lies in the plane C^1_w in either of two circular quadrangles which are obtained by taking the intersection of the annuli Γ_1 and $\{r_2 < |w - 1| < 1\}$, while the point z_0 lies in the plane C^1_z in either of two circular quadrangles which are obtained by taking the intersection of the annuli Γ_2 and $\{r_1 < |z + 1| < 1\}$. In order to obtain domains in which the parts of $\Gamma \cap E$ lie, we have to multiply quadrangles lying in the upper

half-plane on the one hand and in the lower half-plane on the other. We denote by Σ the component of the intersection $\Gamma \cap E$ contained in the former of these domains.

In the domain Γ we construct a meromorphic function $g(w, z)$, equivalent with respect to subtraction to the function $1/(w - z - 1)$ at points $(w, z) \in \Sigma$ and holomorphic at points $(w, z) \notin \Sigma$. In view of Cousin's first theorem (Theorem 25.1, (I)), such a function exists for polycylindrical domains. Next we form an open set B consisting of those points of the domain Γ for which $|g(w, z)| < M$. Here $M > 0$ is some number. By Theorems 11.7, (I), 11.8, (I) and 12.7, (I) its connected components are domains of holomorphy.

We choose a number d such that $0 < d < (1 - r_2)/6$ and associate with each point $(\omega, \zeta) \in \Sigma$ the disk $\gamma_{(\omega, \zeta)} = \{|z - \zeta| < d,\ w = \omega\}$. Further we form sets: 1) $G_d = \bigcup_{(\omega, \zeta) \in \Sigma} \gamma_{(\omega, \zeta)}$, 2) $F_d = \{r_2 + d < |z| < 1 - d,\ |w| = (1 + r_1)/2\} \setminus G_d$. Evidently the function $g(w, z)$ is holomorphic at points of the set $\overline{F}_d$. We choose the number M so large that $B \supset \overline{F}_d$. Let the domain B_0 be the connected component of the set B which contains the set $\overline{F}_d$.

We shall show (by contradiction) that the domain of holomorphy B_0 fails to be a Runge domain of the second kind. Indeed, otherwise, the function $g(w, z)$ in an arbitrary domain $D \supset \overline{F}_d$, where $\overline{D} \subset B_0$, could be represented as a limit of uniformly convergent sequence of rational functions $\{f_k(w, z),\ k = 1, 2, \cdots\}$ holomorphic in the domain B_0.

From the circle $S = \{|w| = (1 + r_1)/2\} \subset \Gamma_1$ we remove an arc σ. We agree to say that a point $\omega \in \sigma$ if the point $(\omega, \zeta) \in \Sigma$ and the disk $\{|z - \zeta| < d\} \subset \{r_2 + d < |z| < 1 - d\}$. The arc σ is not empty because of the condition on the choice of the number d. Further, we consider at points $\omega \in \sigma$ a sequence of rational functions

$$(H) \qquad \{f_k(\omega, z),\quad k = 1, 2, \cdots\}.$$

We assert that among these functions at least one has poles in the disk $|z - \zeta| < d$. In fact, all the functions $f_k(w, z)$ are holomorphic on the circle $\{w = \omega,\ |z - \zeta| = d\} \subset F_d$ and the sequence (H) converges uniformly to the function $g(w, z)$ on the same circle. If all the functions $f_k(\omega, z)$ were holomorphic in the disk $|z - \zeta| < d$, then the limit function $g(\omega, z)$ must also be holomorphic there. However, by definition, this last function has a pole at the center of the disk, namely, at the point (ω, ζ). Thus our assertion is proved.

Suppose that the function $f_m(w, z)$ has poles in the disk $|z - \zeta| < d$. We put

$$f_m(w, z) = \frac{\phi(w, z)}{\psi(w, z)},$$

where $\phi(w, z)$ and $\psi(w, z)$ are mutually prime polynomials. Consider the equation $\psi(w, z) = 0$ at various points $w \in S$. If $w = \omega \in \sigma \subset S$, it has roots in the disk $|z - \zeta| < d$. If we move the point $w = \omega$ along the arc σ these roots never leave the disk, since on the circle $\{|z - \zeta| = d, \omega \in \sigma\} \subset F_d$ the function $f_m(\omega, z)$ is holomorphic and the polynomial $\psi(\omega, z)$ has no roots. Therefore the number of such roots (each root is enumerated as many times as its multiplicity) in this disk is constant. We denote it by λ.

We now consider the number of roots of the equation $\psi(w, z) = 0$ in the disk $|z| < (1 + r_2)/2$. Let us make the point w go around the circle S in the positive direction. Taking into account that the polynomial $\psi(w, z) \neq 0$ at points $(w, z) \in F_d$, we establish: when the disk $|z - \zeta| < d$ leaves the disk $|z| < (1 + r_2)/2$ as the point $w = \omega \in \sigma$ moves, λ roots of the equation $\psi(w, z) = 0$ also leave this disk. As the point w moves further along the circle S, the number of roots under consideration remains unchanged. We arrive at a contradiction, since after a full revolution of the circle S the point w returns to its initial position.

Hence the domain of holomorphy B_0 is not a Runge domain of the second kind. Thus we have proved

THEOREM 3.3. *In the space C^n for $n > 1$ there exist domains of holomorphy which are neither Runge domains of the first kind nor of second kind.*

We note that in the space C^3 there exist domains which can be mapped biholomorphically onto the unit polycylinder and are not Runge domains of the second kind.[1)]

We now formulate a theorem of Serre. It contains a necessary (but not sufficient) condition for a domain of holomorphy to be a Runge domain of the first kind.

THEOREM 3.4 (Serre [2]). *The Betti numbers of dimension greater than or equal to n are equal to zero for a domain of holomorphy $D \subset C^n$ which is a Runge domain of the first kind.*

This theorem is an immediate generalization of Runge's theorem. For its proof see §12.5, Chapter II.

We shall further note the following result (Behnke-Stein [3]): *Every star-shaped*

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1) See G. Stolzenberg, *An example concerning rational convexity*, Math. Ann. 147 (1962), 275–276.

domain of holomorphy $D \subset C^n$ *is a Runge domain of the first kind* (see the following subsection).

3. Runge domain relative to an embracing domain. If D^* is a domain over the space C^n and D is a subdomain of the domain D^*, then the ring $\mathfrak{O}_{D^*}$ of functions holomorphic in D^* is a complete family of functions holomorphic in the domain D.

DEFINITION (holomorphic convexity of a domain D relative to a domain D^*). Let D and D^* be domains of holomorphy over the space C^n. The domain D is said to be *holomorphically convex relative to the domain* D^* if it is strongly convex relative to the family $\mathfrak{O}_{D^*}$ of functions holomorphic in the domain D^*.

By Theorem 2.2 the relation $D\Re D^*$ holds if and only if the domain D (which was assumed to be finite-sheeted in Theorem 2.2) is holomorphically convex relative to the domain D^*.

The concept of a holomorphic enlargement of a domain plays an important role in the theory of Runge domains.

DEFINITION (holomorphic enlargement of a domain). Let D and D^* be domains of holomorphy over the space C^n, the domain D being a subdomain of the domain D^*. The domain D can be holomorphically enlarged to the domain D^* if, for all domains $\mathfrak{D} \Subset D$, $\mathfrak{D}^* \Subset D^*$ and for every number $\epsilon > 0$, one can find domains of holomorphy $D^1, \cdots, D^N \subset D^*$ possessing the properties: 1) $\mathfrak{D} \Subset D^1 \subset D$; 2) $\mathfrak{D}^* \Subset D^N \subset D^*$; 3) $D^1 \subset \cdots \subset D^N$; 4) $\hat{D}^j_\epsilon \Subset D^{j-1}$, $j = 2, \cdots, N$.

Here $D^j_\epsilon = \{z \in D^j,\ d_{D^j}(z) > \epsilon\}$, and $d_{D^j}(z)$ is the boundary distance of the point z in the domain D^j (if D^* is an unbounded domain this distance is taken in the chordal metric, see §5, Chapter I, (I)), and $\hat{D}^j_\epsilon$ is the $\mathfrak{O}_{D^j}$-convex hull of the set D^j_ϵ (see §11.2, Chapter II, (I)).

The domains $D^1, \cdots, D^N$ are called an *ϵ-chain for the holomorphic enlargement of the domain D to the domain D^*.*

EXAMPLES. 1) The annulus $1/2 < |z| < 1$ on the plane C^1 of the variable z can be holomorphically enlarged, to the disk, except for its center, $0 < |z| < 1$, but not to the entire unit disk; an ϵ-chain of that enlargement can be composed of the annuli $\{t < |z| < 1\}$. Values of t from the interval $0 < t < 1/2$ are to be taken frequently enough to ensure the fulfillment of the conditions 1)–4) in the definition of holomorphic enlargement.

2) On a closed simply-connected Riemann surface over the plane C^1_z the single-sheeted disk can be holomorphically enlarged to the entire surface, with

one exceptional point.

3) It is impossible to enlarge holomorphically the bounded domain of holomorphy

$$\{|w| < 1, \ |z| < 1, \ z \neq \phi(w), \ w \neq \phi(z)\}$$

of the space $C^2_{w,z}$. Here $\phi(z)$ is a modular function defined in the unit disk $|z| < 1$.

We can now prove the following theorem (Behnke-Stein [2], (I), Stein [1]):

THEOREM 3.5. *Let D and D^* be finite-sheeted domains of holomorphy over the space C^n_z, while $D \subset D^*$. Then the domain D is holomorphically convex relative to the domain D^* if and only if the domain D can be holomorphically enlarged to the domain D^*.*

REMARK. In view of Theorem 2.2, Theorem 3.5 provides a necessary and sufficient condition for the relation $D\Re D^*$. If D and D^* are not domains of holomorphy, these should be replaced in this definition by their holomorphy hulls $H(D)$ and $H(D^*)$.

PROOF. Sufficiency of the condition given in Theorem 3.5 follows from the following lemma (we shall omit the proof of its necessity):

LEMMA. *Let G_1, G_2, G_3 be finite-sheeted domains over the space C^n_z, and let G_3 be a domain of holomorphy such that $\hat{G}_1 \Subset G_2 \subset G_3$, where $\hat{G}_1$ is the $\mathfrak{O}_{G_3}$-convex hull of the domain G_1. Then any function holomorphic in the domain G_2 can be represented in the domain G_1 by a uniformly convergent series which consists of functions holomorphic in the domain G_3.*

PROOF OF THE LEMMA. The domain of holomorphy G_3 is holomorphically convex relative to the family of functions $\mathfrak{O}_{G_3}$. Reasoning in the same way as in the proof of Theorem 2.1 (see §2.2) we can show that $\hat{G}_1 \Subset G_2 \subset G_3$ and that there exists a Weil analytic polyhedron Δ, defined by the functions $f_j(z) \in \mathfrak{O}_{G_3}$, such that $G_1 \Subset \Delta \Subset G_3$. By making use of Lemma 10 in the preceding section we establish that any function $F(z) \in \mathfrak{O}_{G_2}$ can be represented in the domain G_1 by a uniformly convergent series which consists of functions holomorphic in the domain G_3. The lemma is proved.

Now we turn to the direct proof of Theorem 3.5. We choose a number ϵ, which defines an ϵ-chain for the holomorphic enlargement of the domain D to the domain D^*, so small that $\mathfrak{D} \Subset D^1_\epsilon$ and $\mathfrak{D}^* \Subset D^N_\epsilon$. First we apply the above lemma to the domains $[D^2_\epsilon]$, D^1, D^2 (where $[D^2_\epsilon]$ is the connected component of the set D^2_ϵ which contains the domain $\mathfrak{D}$), and then to the domains $[D^3_\epsilon]$, D^2, D^3, and so forth. After $N-1$ steps, the assertion of Theorem 3.5 will be proved.

DEFINITION (semicontinuous enlargement of a domain). Let D and D^* be domains over the space C_z^n, $D \subset D^*$. The domain D can be semicontinuously enlarged to the domain D^* if there exists a set of domains D_s $(0 \leq s \leq 1)$ over the space C_z^n having the following properties:

1) $D_0 = D$, $D_1 = D^*$;

2) $D_{s_1} \subset D_{s_2}$ for $s_1 < s_2$;

3) any limit point of an arbitrary sequence of points $z_\nu \in \partial D_{s_\nu}$, where $s_\nu \leq s_0$, $0 \leq s_0 \leq 1$, $\nu = 1, 2, \cdots$, $\lim_{\nu \to \infty} s_\nu = s_0$, lies on the boundary ∂D_{s_0};

4) every point of the boundaries ∂D_{s_0}, $s_0 < 1$, is a limit point for a certain sequence of points $z_\nu \in \partial D_{s_\nu}$, where $s_\nu > s_0$, $\nu = 1, 2, \cdots$, $\lim_{\nu \to \infty} s_\nu = s_0$.

It is clearly easy to establish that the domain D^* can be obtained from the domain D by a semicontinuous enlargement, rather than by a holomorphic enlargement. Therefore the following theorem is of practical use:

THEOREM 3.6. *If a domain of holomorphy D over the space C_z^n can be semicontinuously enlarged to the domain D^* over the space C_z^n, then it is possible to enlarge holomorphically the domain D to the domain D^*.*

The proof of this theorem will be omitted. It consists in selecting from a set of domains D_s $(0 \leq s \leq 1)$, by means of which the semicontinuous enlargement is realized, an ϵ-chain of domains employed for the holomorphic enlargement.

It is easy to see that any star-shaped domain $D \subset C^n$ can be semicontinuously enlarged to the entire space C^n. Hence it follows in view of Theorems 3.5 and 3.6 that the domain D is convex relative to the entire function. Since the entire function can be uniformly approximated in the domain D by a polynomial, we conclude that D is a Runge domain of the first kind.

4. **Approximation of functions on complex manifolds and spaces.** We consider a holomorphically complete complex space $\mathfrak{R}^*$ consisting of an at most countable set of connected components (see §18.3, Chapter III, (I)), a domain $\mathfrak{R} \subset \mathfrak{R}^*$ and the product $\mathfrak{R}^* \times \mathfrak{R}^*$ of two copies of the space $\mathfrak{R}^*$. By ∇ we denote some neighborhood of the "diagonal" of the product $\mathfrak{R}^* \times \mathfrak{R}^*$ (this diagonal consists of points (r, r) with $r \in \mathfrak{R}^*$), and by $U^\nabla(r_0)$, where $r_0 \in \mathfrak{R}$, a set of those points $r \in \mathfrak{R}^*$ for which $(r_0, r) \in \nabla$. Let $\mathfrak{R}_\nabla = \{r \in \mathfrak{R}, U^\nabla(r) \subset \mathfrak{R}\}$.

DEFINITION (holomorphic enlargement of a domain $\mathfrak{R}$ to a holomorphically complete space $\mathfrak{R}^*$). A holomorphically convex domain $\mathfrak{R}$ can be holomorphically enlarged to the space $\mathfrak{R}^*$ if for all compact sets $K \subset \mathfrak{R}$, $K^* \subset \mathfrak{R}^*$ and for the

neighborhood ∇ one can find holomorphically convex domains $\mathfrak{R}^1, \cdots, \mathfrak{R}^N \subset \mathfrak{R}^*$ having the following properties: 1) $K \subset \mathfrak{R}^1 \subset \mathfrak{R}$; 2) $K^* \subset \mathfrak{R}^N \subset \mathfrak{R}^*$; 3) $\mathfrak{R}^1 \subset \cdots \cdots \subset \mathfrak{R}^N$; 4) $\hat{\mathfrak{R}}^j_\nabla \Subset \mathfrak{R}^{j+1}$, $j = 2, \cdots, N$.

Here $\hat{\mathfrak{R}}^j_\nabla$ is the holomorphically convex hull of the set $\mathfrak{R}^j_\nabla$ (see §18.3, Chapter III, (I)). For domains over the space C^n this definition reduces to the one given in the preceding subsection.

Grauert [2] established the following theorem:

THEOREM 3.7. *Let $\mathfrak{R}$ and $\mathfrak{R}^*$ be holomorphically complete complex spaces, while the space $\mathfrak{R}$ is contained in $\mathfrak{R}^*$ and can be holomorphically enlarged to it. Then every function holomorphic on the space $\mathfrak{R}$ can be represented there by a uniformly convergent series which consists of functions holomorphic on the entire space $\mathfrak{R}^*$.*

Further, Will [1] obtained a stronger result for holomorphically complete manifolds $\mathfrak{R}$ and $\mathfrak{R}^*$: he proved that in this case the conditions contained in Theorem 3.7 are sufficient as well as necessary. [1)]

By using the concept of a neighborhood ∇ of the diagonal of the product $\mathfrak{R} \times \mathfrak{R}$ of two copies of a complex space $\mathfrak{R}$, we can formulate the general concept of the uniform convergence of a sequence of mappings $F_\nu: \mathfrak{T} \longrightarrow \mathfrak{R}$, $\nu = 1, 2, \cdots$, to a mapping $F: \mathfrak{T} \longrightarrow \mathfrak{R}$. Here $\mathfrak{T}$ is a certain topological space.

DEFINITION (uniform convergence of mappings). A sequence of mappings F_ν, $\nu = 1, 2, \cdots$, converges uniformly on the space $\mathfrak{T}$ to a mapping F if for every compact $K \subset \mathfrak{T}$ and for every neighborhood ∇ of the diagonal one can find a number ν_0 such that for $\nu \geq \nu_0$ and $t \in K$

$$(F(t),\ F_\nu(t)) \in \nabla.$$

We note that if $\mathfrak{R}$ is a domain in the space C^n_z and if $\mathfrak{T}$ is a domain in the space C^l_t, and if the mappings F_ν and F are given by a set of functions

$$(F_\nu) \qquad z_k^{(\nu)} = f_k^{(\nu)}(t_1, \ldots, t_l), \qquad \nu = 1, 2, \ldots,$$

$$(F) \qquad z_k = f_k(t_1, \ldots, t_l).$$

then the uniform convergence of the sequence of mappings $\{F_\nu\}$ to the mapping F is reduced to the uniform convergence of the functions $f^{(\nu)}_k$ to the functions f_k. In the case that $\mathfrak{R}$ and $\mathfrak{T}$ are domains over the spaces C^n_z and C^l_t, the functions $f^{(\nu)}_k$ and f_k must be given and must define points on the elements of corresponding

1) See also Theorem 11.5, Chapter II.

coverings. The limit of a uniformly convergent sequence of holomorphic mappings is always a holomorphic mapping.

We note that instead of a mapping of the space $\mathfrak{T}$ onto the space $\mathfrak{R}$ we often speak of a *function defined on the space* $\mathfrak{T}$ *and assuming its values in the space* $\mathfrak{R}$. If both spaces $\mathfrak{R}$ and $\mathfrak{T}$ are complex and the mapping $\mathfrak{T} \longrightarrow \mathfrak{R}$ is holomorphic, then we speak of a *holomorphic function given on the complex space* $\mathfrak{T}$ *and assuming its values in the complex space* $\mathfrak{R}$.

We shall formulate a general problem of the theory of approximation.

PROBLEM. *Let* $\mathfrak{R}$ *and* $\mathfrak{R}^*$ *be holomorphically complete complex spaces, while the space* $\mathfrak{R}$ *is contained in the space* $\mathfrak{R}^*$ *and can be holomorphically enlarged to it, and let* F *be some holomorphic function defined on the space* $\mathfrak{R}$ *and assuming its values in the complex space* $\mathfrak{W}$. *Under what conditions does there exist a sequence of holomorphic functions* $F_\nu: \mathfrak{R}^* \longrightarrow \mathfrak{W}$, $\nu = 1, 2, \cdots$, *which converges uniformly on the space* $\mathfrak{R}$ *to the function* F?

In its general form this problem is still far from being settled. Essential results in the cases when $\mathfrak{W}$ is a Riemann surface or a complex Lie group (for the definition, see §10, Chapter II) were obtained by H. Grauert [2,3], and for holomorphically complete complex manifolds by Ramspott and Stein [1].

§4. EXPANSION BY ORTHOGONAL FUNCTIONS

1. Orthonormal systems. Expansions of functions into power series of various forms in a disk-shaped domain are special cases of orthogonal function expansions in an arbitrary domain D of the space C^n. Such expansions are of great importance in the theory of biholomorphic mappings. [1)]

Let D be a domain of the space C^n of the variables $z_1, \cdots, z_n$. We consider the collection (often called the class) $L^2(D)$ of functions $f(z)$ holomorphic in the domain D, for which the Lebesgue integral is finite:

$$\|f\|_D^2 = \int_D |f|^2 \, dv < \infty. \tag{1.32}$$

Here dv is the volume element in the space C^n. If necessary, the integral (1.32) is to be understood as improper, i.e., as $\lim_{\nu\to\infty} \int_{D_\nu} |f|^2 \, dv$, where $\{D_\nu, \nu = 0, 1, 2, \cdots\}$ is a principal sequence of domains which approximates the domain D from

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1) The theory of orthogonal functions of several complex variables has been constructed by S. Bergman [1,2,5] and by S. Bochner [1]. See also the books by S. Bergman [9–11].

the inside. In what follows the quantity $\|f\|_D$ is called the *norm* of the function $f(z)$ in the domain D.

DEFINITION (inner product of functions). Suppose that functions $f, g \in L^2(D)$. The quantity

$$(f, g)_D = \int_D f\bar{g}\, dv \tag{1.33}$$

is called the *inner product* of the functions f and g in the domain D. If necessary the integral (1.33) is also to be understood as improper.

It is easy to prove (by using the Bunjakovskiĭ-Schwarz inequality) that the inner product of the functions $f, g \in L^2(D)$ always exists.

Evidently $(f, f)_D = \|f\|_D^2$.

DEFINITION (orthonormal system of functions). A function $f \in L^2(D)$ is said to be *normalized* in the domain D if $\|f\|_D = 1$. Functions $f, g \in L^2(D)$ are said to be *orthogonal* in the domain D if $(f, g)_D = 0$. A finite or countable set of functions ϕ_ν $(\nu = 0, 1, 2, \cdots)$ is said to be orthonormal in the domain D if

$$(\phi_\nu, \phi_\mu)_D = \begin{cases} 0 & \text{for } \nu \neq \mu, \\ 1 & \text{for } \nu = \mu. \end{cases} \tag{1.34}$$

Examples of orthonormal system will be derived below (see subsection 4 of the present section).

Together with the domain D in the space C_z^n of the variables $z_1, \cdots, z_n$, we shall consider the same domain D in the space C_ζ^n of variables $\zeta_1, \cdots, \zeta_n$. We denote these domains by D_z and D_ζ, respectively. Further, we denote by $D_{\bar{z}}$ an image of the domain D under a mapping $(z_1, \cdots, z_n) \longrightarrow (\bar{z}_1, \cdots, \bar{z}_n)$; this mapping will be denoted simply by the symbol $z \longrightarrow \bar{z}$.

THEOREM 4.1. *If $\{\phi_\nu(z), \nu = 0, 1, 2, \cdots\}$ is an orthonormal system of functions in a bounded domain $D \subset C_z^n$, then the series*

$$\sum_{\nu=0}^{\infty} \varphi_\nu(z)\overline{\varphi_\nu(\zeta)}$$

converges absolutely and uniformly in the domain $D_z \times D_\zeta$ of the space $C_{z,\zeta}^{2n}$ of the variables $z_1, \cdots, z_n, \zeta_1, \cdots, \zeta_n$.

This theorem results immediately from the following lemma:

LEMMA. *If the function $f(z)$ is holomorphic in the polycylinder $E = \{|z_j - a_j| < r_j,\ j = 1, \cdots, n\}$, then*

$$\pi^{\frac{n}{2}} r_1 \ldots r_n |f(a_1, \ldots, a_n)| \leqslant \|f\|_E. \tag{1.35}$$

PROOF. We first assume that the function f is holomorphic in the closed polycylinder $\overline{E}$ and can thus be represented there by a uniformly convergent series

$$f(z)=\sum_k c_k (z-a)^k, \tag{1.36}$$

which allows termwise integration. Here, as usual $c_k = c_{k_1 \dots k_n}$, $(z-a)^k = (z_1-a_1)^{k_1} \cdots (z_n-a_n)^{k_n}$, and the summation with respect to k is taken over $k_1, \cdots, k_n = 0, 1, 2, \cdots$.

Substituting the series (1.36) into the expression for the norm (1.32) and passing in each z_j-plane to the polar coordinates ρ_j, θ_j $(j = 1, \cdots, n)$ with their center at the point $(a_1, \cdots, a_n)$, we obtain

$$\begin{aligned} \|f\|_E^2 = & \sum_{\substack{k_i, l_i=0 \\ i=1, \dots, n}}^{\infty} c_{k_1 \dots k_n} \overline{c_{l_1 \dots l_n}} \\ & \times \prod_{j=1}^{n} \int_0^{r_j} \int_0^{2\pi} \rho_j^{k_j+l_j+1} e^{i(k_j-l_j)\theta_j} d\rho_j d\theta_j \\ = \pi^n & \sum_{k_1, \dots, k_n=0}^{\infty} \left(\prod_{j=1}^{n} (k_j+1)^{-1} \right) |c_{k_1 \dots k_n}|^2 r_1^{2k_1+2} \dots r_n^{2k_n+2} \\ & \geqslant \pi^n r_1^2 \dots r_n^2 |c_{0 \dots 0}|^2 = \pi^n r_1^2 \dots r_n^2 |f(a_1, \dots, a_n)|^2. \end{aligned}$$

Thus we get the inequality (1.35) for the function $f(z)$ holomorphic in the closed polycylinder $\overline{E}$. If the function $f(z)$ is holomorphic only in the open polycylinder E, we first write the inequality (1.35) for the polycylinder

$$E_\epsilon = \{ |z_j - a_j| < r_j - \epsilon, \ j = 1, \dots, n \}$$

and then pass to the limit as $\epsilon \longrightarrow 0$. As a result we obtain the inequality (1.35) for the open polycylinder.

We now turn back to the proof of Theorem 4.1. Suppose that the point $a \in D$ has the coordinates $a_1, \cdots, a_n$; let us choose the numbers $r_1, \cdots, r_n$ in such a way that the closed polycylinder $\overline{E} \subset D$. Then we have

$$\begin{aligned} \sum_{\nu=1}^{m} |\varphi_\nu(a)|^2 = \int_D \left| \sum_{\nu=1}^{m} \varphi_\nu(z) \overline{\varphi_\nu(a)} \right|^2 dv \\ \geqslant \int_E \left| \sum_{\nu=1}^{m} \varphi_\nu(z) \overline{\varphi_\nu(a)} \right|^2 dv. \end{aligned}$$

In view of the above lemma the last integral is greater than

$$\pi^n r_1^2 \cdots r_n^2 \left[\sum_{\nu=1}^{m} |\phi_\nu(a)|^2 \right]^2.$$

Replacing it by this expression we find that

$$\sum_{\nu=0}^{m} |\varphi_\nu(a)|^2 < \frac{1}{\pi^n r_1^2 \ldots r_n^2}.$$

Now, in view of the Bunjakovskiĭ-Schwarz inequality we have

$$\left(\sum_{\nu=1}^{m} |\varphi_\nu(z)\overline{\varphi_\nu(\zeta)}|\right)^2 \leqslant \left(\sum_{\nu=1}^{m} |\varphi_\nu(z)|^2\right)\left(\sum_{\nu=1}^{m} |\varphi_\nu(\zeta)|^2\right) \leqslant \frac{1}{\pi^{2n} r_1^2 \ldots r_n^2 \rho_1^2 \ldots \rho_n^2}.$$

Here the numbers $r_1, \cdots, r_n, \rho_1, \cdots, \rho_n$ are so chosen that the closed polycylinders $\{|Z_j - z_j| \leq r_j,\ j = 1, \cdots, n\}$ and $\{|Z_j - \zeta_j| \leq \rho_j,\ j = 1, \cdots, n\}$ lie inside the domain D.

Hence, because of the boundedness of the domain D, the assertion of the theorem easily follows.

DEFINITION (kernel function of an orthonormal system of functions). The quantity

$$\sum_{\nu=0}^{\infty} \varphi_\nu(z)\overline{\varphi_\nu(\zeta)} = K(z, \overline{\zeta}) \tag{1.37}$$

is called the kernel function of the orthonormal system of functions $\{\phi_\nu(z),\ \nu = 0, 1, 2, \cdots\}$ in the domain D.

From Theorem 4.1 then follows the

COROLLARY. *$K(z, \overline{\zeta})$ is a holomorphic function of the variables $z_1, \cdots \cdots, z_n, \overline{\zeta}_1, \cdots, \overline{\zeta}_n$ in the domain $D_z \times D_{\overline{\zeta}}$ of the space $C^{2n}_{z,\overline{\zeta}}$.*

2. Closed systems. Let $\{\phi_\nu(z),\ \nu = 0, 1, 2, \cdots\}$ be an orthonormal system of functions in a bounded domain D, and let a function $f(z) \in L^2(D)$. The quantities

$$a_\nu = (f, \phi_\nu)_D \tag{1.38}$$

are called the *Fourier coefficients* of the function $f(z)$ with respect to the orthonormal system $\{\phi_\nu(z),\ \nu = 0, 1, 2, \cdots\}$.

If b_ν $(\nu = 0, 1, 2, \cdots)$ are arbitrary numbers, then it is easy to see that

$$M_m^2 = \left\| f - \sum_{\nu=0}^{m} b_\nu \varphi_\nu \right\|_D^2 = \| f \|_D^2 + \sum_{\nu=1}^{m} | b_\nu - a_\nu |^2 - \sum_{\nu=1}^{m} | a_\nu |^2. \tag{1.39}$$

From this equation it follows that:

1) The quantity M_m attains its maximum when all $b_\nu = a_\nu$. In this sense the sum $\Sigma_{\nu=1}^{m} a_\nu \phi_\nu$ is the best approximation to the function in the sense of the norm $\| \cdots \|_D$.

2) *Bessel's inequality* is valid:

$$\sum_{\nu=0}^{\infty} | a_\nu |^2 \leqslant \| f \|_D. \tag{1.40}$$

THEOREM 4.2. *If a_ν are Fourier coefficients of some function $f \in L^2(D)$ with respect to an orthonormal system of functions $\{\phi_\nu(z),\ \nu = 0, 1, 2, \cdots\}$ in a bounded domain D, then the series $\Sigma_{\nu=0}^{\infty} a_\nu \phi_\nu$ converges absolutely and uniformly in this domain D.*

PROOF. In view of the Bunjakovskiĭ-Schwarz inequality it follows from the relations (1.37) and (1.40) that

$$\left(\sum_{\nu=0}^{m} | a_\nu \varphi_\nu (z) | \right)^2 \leqslant \sum_{\nu=0}^{m} | a_\nu |^2 \sum_{\nu=0}^{m} | \varphi_\nu (z) |^2 \leqslant \| f \|_D K(z, \bar{z}).$$

Hence there follows the absolute convergence of the series under consideration. In order to establish its uniform convergence, we again apply the Bunjakovskiĭ-Schwarz inequality. We find that

$$\left(\sum_{\nu=m+1}^{\infty} | a_\nu \varphi_\nu | \right)^2 \leqslant \sum_{\nu=m+1}^{\infty} | a_\nu |^2 \sum_{\nu=m+1}^{\infty} | \varphi_\nu |^2 \leqslant \| f \|_D \sum_{\nu=m+1}^{\infty} | \varphi_\nu |^2.$$

Hence, in view of the uniform convergence of the series (1.37) in the domain D, our second assertion follows.

DEFINITION (closed system of functions). An orthonormal *system of functions* $\{\phi_\nu (z),\ \nu = 0, 1, 2, \cdots\}$ is said to be *closed* in the domain D if for every function $f(z) \in L^2(D)$ it happens that (instead of (1.40))

$$\sum_{\nu=0}^{\infty} | a_\nu |^2 = \| f \|_D. \tag{1.41}$$

The relation (1.41) is called *Parseval's equality*.

From equation (1.39) it follows that for the closed system of functions

$\{\phi_\nu(z),\ \nu = 0, 1, 2, \cdots\}$ and the function $f \in L^2(D)$ we shall have that $\lim_{m\to\infty} M_m = 0$ for $b_\nu = a_\nu$ $(\nu = 0, 1, 2, \cdots)$. In this case Theorem 4.2 can be complemented by the following proposition:

THEOREM 4.3. *Let $\{\phi_\nu,\ \nu = 0, 1, 2, \cdots\}$ be a closed orthonormal system of functions in a bounded domain D, and let a_ν be the Fourier coefficients of a function $f \in L^2(D)$ with respect to this system. Then for $z \in D$*

$$\sum_{\nu=0}^{\infty} a_\nu \varphi_\nu(z) = f(z). \tag{1.42}$$

PROOF. Suppose that the point z belongs to a domain D_0, where the closed domain $\overline{D}_0 \subset D$, and let ρ be the minimal boundary distance of the domain D_0 in the domain D. Then the polycylinder $E_\rho = \{|Z_j - z_j| < \rho,\ j = 1, \cdots, n\} \subset D$ and by the lemma to Theorem 4.1 we have

$$\left| f(z) - \sum_{\nu=1}^{m} a_\nu \varphi_\nu(z) \right|^2 \leqslant \frac{1}{\pi^n \rho^{2n}} \left\| f - \sum_{\nu=1}^{m} a_\nu \varphi_\nu \right\|_{E_\rho}^2$$

$$\leqslant \frac{1}{\pi^n \rho^{2n}} \left\| f - \sum_{\nu=1}^{m} a_\nu \varphi_\nu \right\|_D^2 = \frac{1}{\pi^n \rho^{2n}} \left(\|f\|_D^2 - \sum_{\nu=1}^{m} |a_\nu|^2 \right). \tag{1.43}$$

We note that the last equality comes from the formula (1.39). We have already shown above that in this case $\lim_{m\to\infty} M_m = 0$. Therefore the assertion of the theorem follows from the relation (1.43).

We now prove a theorem which is in some sense a converse of the preceding theorem.

THEOREM 4.4. *If $\{\phi_\nu,\ \nu = 0, 1, 2, \cdots\}$ is an orthonormal system of functions in a bounded domain D and every function $f \in L^2(D)$ can be expressed by a series*

$$f(z) = \sum_{\nu=0}^{\infty} b_\nu \varphi_\nu(z), \tag{1.44}$$

which converges uniformly in this domain D, then the system of functions $\{\phi_\nu,\ \nu = 0, 1, 2, \cdots\}$ is closed. Here it is assumed that $\sum_{\nu=0}^{\infty} |b_\nu|^2 < \infty$.

REMARK. The last requirement will be automatically satisfied (in view of (1.40)) for $b_\nu = a_\nu = (f, \phi_\nu)_D$.

PROOF. First we shall show that under the above conditions the series (1.44) will "converge to the function $f(z)$ in the sense of the norm", i.e., that

$$\lim_{m\to\infty}\left\| f-\sum_{\nu=0}^{m} b_\nu\varphi_\nu \right\|_D = \lim_{m\to\infty}\left\| \sum_{\nu=m+1}^{\infty} b_\nu\varphi_\nu \right\|_D = \lim_{m\to\infty}\|r_m\|_D = 0. \tag{1.45}$$

Let

$$r_{mp}(z) = \sum_{\nu=m+1}^{m+p} b_\nu\varphi_\nu(z), \qquad p=1,\ 2,\ \ldots.$$

Evidently $\lim_{p\to\infty} r_{mp}(z) = r_m(z)$, and this limit is approached uniformly in the domain D. Therefore for $z \in D_0$, where the closed domain $\overline{D}_0 \subset D$, we have

$$\|r_m\|_{D_0}^2 = \lim_{p\to\infty}\|r_{mp}\|_{D_0}^2 \leqslant \lim_{p\to\infty}\|r_{mp}\|_D^2 = \sum_{\nu=m+1}^{\infty} |b_\nu|^2.$$

Hence it follows that

$$\|r_m\|_D \leqslant \sum_{\nu=m+1}^{\infty} |b_\nu|^2.$$

But $\lim_{m\to\infty}\sum_{\nu=m+1}^{\infty}|b_\nu|^2 = 0$ because of the convergence of the series $\sum_{\nu=0}^{\infty}|b_\nu|^2$. Therefore $\lim_{m\to\infty}\|r_m(z)\|_D = 0$.

We now establish that $b_k = (f, \phi_k)_D = a_k$. Indeed, for $m \longrightarrow \infty$

$$|(f, \varphi_k)_D - b_k|^2 = \left|\left(f-\sum_{\nu=0}^{m} b_\nu\varphi_\nu,\ \varphi_k\right)_D\right|^2$$
$$\leqslant \left\| f-\sum_{\nu=0}^{m} b_\nu\varphi_\nu \right\|_D \|\varphi_k\|_D = \left\| f-\sum_{\nu=0}^{m} b_\nu\varphi_\nu \right\|_D \to 0.$$

Thus $b_\nu = a_\nu$. Then from the relations (1.39) and (1.45) it follows that in our case

$$\|f\|_D^2 = \sum_{\nu=0}^{\infty} |a_\nu|^2.$$

With this our theorem is proved.

SUPPLEMENTARY REMARK. By the method used for the proof of Theorem 4.4 we can show that for $\sum_{\nu=0}^{\infty}|b_\nu|^2 < \infty$, $\sum_{\nu=0}^{\infty}|c_\nu|^2 < \infty$ the quantity $(\sum_{\nu=0}^{\infty} b_\nu\phi_\nu,\ \sum_{\nu=0}^{\infty} c_\nu\phi_\nu)$ can be found by termwise integration. We shall use this result below.

Now we turn to the proof of a fundamental proposition of the theory which we are studying.

THEOREM 4.5. *In every bounded domain $D \subset C^n$ there exists a closed orthonormal system of functions.*

PROOF. I. Suppose that a point $a(a_1, \cdots, a_n)$ is in D and that E_ν is a

set of functions $f \in L^2(D)$ for which $f_{p_1 \cdots p_n} = 0$ for $(p_1 \cdots p_n) < \nu$ and $f_{p_1 \cdots p_n} = 1$ for $(p_1 \cdots p_n) = \nu$. Here

$$f_{p_1 \ldots p_n} = \left.\frac{\partial^{p_1 + \cdots + p_n} f}{\partial z_1^{p_1} \ldots \partial z_n^{p_n}}\right|_{z_j = a_j},$$

and $(p_1 \cdots p_n)$ is the number of the system of natural numbers $p_1 \cdots p_n$ in the sequence

$$00 \ldots 0;\ 10 \ldots 0;\ \ldots;\ 00 \ldots 1;\ 20 \ldots 0;\ 11 \ldots 0;\ \ldots.$$

This sequence is constructed in the following way:

1) First, we combine into a group those systems of numbers $p_1 \cdots p_n$ for which the quantities $p_1 + \cdots + p_n$ are identical; these groups of systems are then arranged in increasing order of the quantity $p_1 + \cdots + p_n$.

2) Then systems of numbers $p_1 \cdots p_n$ within the limits of each group are arranged in the lexicographical order (i.e., like words in the dictionary; here the roles of the letters are played by the places which are occupied by the numbers $0, 1, \cdots, m$ in the systems $p_1 \cdots p_n$ and the number indicates how many times this letter appears).

Evidently the set E_ν is not empty, since for $\nu = (p_1 \cdots p_n)$ the function

$$\frac{(z - a_1)^{p_1} \ldots (z - a_n)^{p_n}}{p_1! \ldots p_n!}$$

belongs to it.

We also consider a variational problem: to find, in the set E_ν, a function which minimizes the integral $\|f\|_D^2$.

Let A be the greatest lower bound of the quantity $\|f\|_D^2$ for functions $f \in E_\nu$; let $\{h_p(z) \in E_\nu,\ p = 0, 1, 2, \cdots\}$ be a sequence of functions for which $\lim_{p \to \infty} \|h_p\|_D^2 = A$. Then the norms $\|h_p\|_D$ are all bounded; hence it follows (see the lemma to Theorem 4.1) that the moduli $|h_p(z)|$ are also all bounded in every domain D_0 with $\overline{D}_0 \subset D$. As is well known,[1] in this case the set of functions $\{h_p(z),\ p = 0, 1, 2, \cdots\}$ is a normal family, and from this sequence we can single out a sequence $\{h_{p_k}(z)\}$ converging uniformly in the domain D. Let

[1] See P. Montel, *Leçons sur les familles normales de fonctions analytiques et leurs applications*, Gauthier-Villars, Paris, 1927, p. 195 of Russian translation. We here use the boundedness of the domain D. For an unbounded domain we could show that $A = \infty$.

$g^{\nu}(z) = \lim_{k\to\infty} h_{p_k}(z)$; evidently $g^{\nu}(z) \in E_{\nu}$ and

$$\|g^{\nu}\|^2_{D_0} = \lim_{k\to\infty} \|h_{p_k}\|^2_{D_0} \leqslant \lim_{k\to\infty} \|h_{p_k}\|^2_{D_0} = A.$$

Then also $\|g^{\nu}\|^2_D \le A$. On the other hand, by definition, $\|g^{\nu}\|^2_D \ge A$. Hence $\|g^{\nu}\|^2_D = A$ and accordingly the function $g^{\nu}(z)$ provides the solution of the variational problem formulated above.

II. We shall show that the function $g^{\nu}(z)$ is orthogonal to every function $\phi(z)$ belonging to $L^2(D)$ and satisfies the condition: $\phi_{p_1\cdots p_n} = 0$ for $(p_1 \cdots p_n) \le \nu$. If $\phi = 0$, then the equality $(g^{\nu}, \phi)_D = 0$ is obvious. If $\phi \not\equiv 0$, then it is easily concluded from the lemma to Theorem 4.1 that $\|\phi\|_D \ne 0$. Indeed, under our assumptions the function $(g^{\nu} + c\phi) \in E_{\nu}$ for any constant c; in particular, for

$$c = c_0 = -(g^{\nu}, \varphi)\,\|\varphi\|_D^{-2}.$$

But

$$\|g^{\nu} + c_0\phi\|^2_D = A - |(g^{\nu}, \phi)_D|^2 \|\phi\|^2_D.$$

Since A is the greatest lower bound of the square of the norm for all functions $f \in E_{\nu}$, the last equality is possible only in the case when $(g^{\nu}, \phi)_D = 0$. Here ϕ is an arbitrary function of the set E_{ν}; thus our assertion is proved.

III. We shall show that the variational problem in question has a unique solution. Indeed, if two functions $g_1^{\nu}(z)$ and $g_2^{\nu}(z)$ satisfy the conditions, then, as was proved in the preceding subsection, $(g_1^{\nu}, g_1^{\nu} - g_2^{\nu})_D = (g_2^{\nu}, g_1^{\nu} - g_2^{\nu})_D = 0$. Thus $\|g_1^{\nu} - g_2^{\nu}\|_D = 0$. By the lemma to Theorem 4.1 it follows that $g_1^{\nu}(z) = g_2^{\nu}(z)$.

IV. Put

$$g_{\nu}(z) = \frac{g^{\nu}(z)}{\|g^{\nu}(z)\|_D}; \tag{1.46}$$

we show that $\{g_{\nu}(z),\ \nu = 0, 1, 2, \cdots\}$ is a closed orthonormal system of functions in the domain D. Its orthonormality comes from II of the present proof and the formula (1.46); we need only to establish that it is closed.

Let $f(z)$ be some function of the class $L^2(D)$. We put $f^{(s)}(z) = \sum_{\nu=0}^{s} c_{\nu}^{(s)} g_{\nu}(z)$, where the coefficients $c_{\nu}^{(s)}$ are so chosen that

$$f^{(s)}_{p_1\cdots p_n} = f_{p_1\cdots p_n} \quad \text{for} \quad (p_1 \cdots p_n) \leqslant s.$$

It is easy to verify directly that such a choice of the quantities $c_{\nu}^{(s)}$ is possible. Because of the orthonormality of the system of functions $\{g_{\nu}(z),\ \nu = 0, 1, 2, \cdots\}$

$$c_\nu^{(s)} = (f^{(s)}, g_\nu)_D.$$

From II of the proof it follows that

$$c = c_0 = -(g^\nu, \varphi)\, \|\varphi\|_D^{-2}.$$

Hence we have

$$\|g^\nu + c_0\varphi\|_D^2 = A - |(g^\nu, \varphi)_D|^2 \|\varphi\|_D^2.$$

We consider the series $\Sigma_{\nu=0}^\infty a_\nu g_\nu(z)$. This series converges absolutely and uniformly in the domain D. Let $\phi(z) = \Sigma_{\nu=0}^\infty a_\nu g_\nu(z)$. Evidently $\phi_{p1\cdots p_n} = f_{p1\cdots p_n}$. But then $f(z) = \phi(z)$ at all points of the domain D.

Thus for any function $f \in L^2(D)$ we have $f(z) = \Sigma_{\nu=0}^\infty a_\nu g_\nu(z)$, where $a_\nu = (f, g_\nu)_D$. Hence, in view of Theorem 4.4, it follows that the system of functions $\{g_\nu(z),\ \nu = 0, 1, 2, \cdots\}$ is closed.

REMARK 1. We note that in the present theorem the assumption of the boundedness of the domain D can be replaced by the weaker requirement of its finiteness. We shall not dwell on the actual realization of the proof for this case.

REMARK 2. The propositions mentioned above can be expressed in a more general fashion in the terminology of the geometry of Hilbert space. In particular, it should be noted that the last theorem is a special case of a general proposition of the theory of this space. However, for our limited aims it is unnecessary to introduce the concepts of Hilbert space. The proof for the existence of the closed system which we have carried out is important for us because it provides an effective method of finding the closed system.

REMARK 3. The method of constructing the closed system of functions given in Theorem 4.5 is not unique. In practice the following method based on Theorem 4.4 is frequently used:

In the domain D we take functions $\psi_\nu \in L^2(D)$, $\nu = 1, 2, \cdots$, having the property that every function $f \in L^2(D)$ can be represented by a series $f(z) = \Sigma_{\nu=1}^\infty b_\nu \psi_\nu(z)$ converging absolutely and uniformly in the domain D. The functions $\psi_\nu(z)$ are, in general, neither assumed to be normalized nor orthonormal to one another, but it is required that an arbitrary finite set of these functions should be linearly independent.

Then, starting from the functions $\psi_\nu(z)$, an orthogonal and normalized system of functions is constructed. For this we form the functions $\phi_1 = a_{11}\psi_1$, $\phi_2 = a_{21}\phi_1 + a_{22}\psi_2, \cdots, \phi_\nu = a_{\nu 1}\phi_1 + \cdots + a_{\nu,\nu-1}\phi_{\nu-1} + a_{\nu\nu}\phi_\nu, \cdots, \cdots$ and

determine the coefficients a_{kl} from the conditions (1.34). It is easily seen that the conditions (1.34) will be satisfied if we take $a_{11} = \|\psi_1\|_D^{-1}$, $a_{21} = - a_{22}(\psi_2, \phi_1)_D$, $a_{22} = \|\psi_2 - (\psi_2, \phi_1)_D \phi_1\|_D^{-1}$ and so forth.

In view of Theorem 4.4 the system of functions $\{\phi_\nu(z),\ \nu = 1, 2, \cdots\}$ is not orthonormal but also closed in the domain D.

We further note that if the functions $\psi_\nu(z)$, $\nu = 1, 2, \cdots$ have the above-mentioned property and can be used for forming a closed orthonormal system of functions, then instead we may take an arbitrary system of functions of the form $\chi_1(z), \cdots, \chi_p(z), \psi_1(z), \psi_2(z), \cdots$. Here $\chi_1(z), \cdots, \chi_p(z)$ are certain linearly independent functions of the class $L^2(D)$. (It is of course clear that for the functions $\psi_\nu(z)$ we can take the functions $g_\nu(z)$ constructed in the proof of Theorem 4.5.)

In particular the first function of the closed orthonormal system can be always chosen arbitrarily apart from the normalization.

3. Kernel function and minimal function of a domain. In general in every bounded domain D there exist an infinity of sets of different closed orthonormal systems of functions. It can be shown that the kernel functions of all such systems coincide with one another and that this kernel function depends only on the domain D. This situation comes from the following theorem:

THEOREM 4.6. *If $\{\phi_\nu(z),\ \nu = 0, 1, 2, \cdots\}$ is a closed orthonormal system of functions in the domain D, then the kernel function of this system*

$$K_D(z_0, \bar{z}_0) = \sum_{\nu=0}^{\infty} |\varphi_\nu(z_0)|^2 \tag{1.47}$$

is the inverse of the minimum of the integral $\|f\|_D^2$ on the set of those functions $f(z) \in L^2(D)$ for which $f(z_0) = 1$. Here z_0 is an arbitrary point of the domain D.

REMARK. In what follows we shall call the collection of functions permissible in the variational problem in Theorem 4.6 "the function class of $L^2(D)$ normalized by the condition $f(z_0) = 1$".

PROOF. By Theorem 4.3 we have

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu \varphi_\nu,$$

where $a_\nu = (f, \phi_\nu)_D$. We set $\phi_\nu(z_0) = \phi_\nu^0$ and define numbers A_ν by

$$a_\nu = \frac{\overline{\varphi_\nu^0} + A_\nu}{K_D(z_0, \bar{z}_0)}.$$

Then

$$f(z)=\frac{\sum_{\nu=0}^{\infty}\left(\overline{\varphi_\nu^0}+A_\nu\right)\varphi_\nu(z)}{K_D(z_0,\bar{z}_0)}. \tag{1.48}$$

If $f(z)$ is a permissible function, then $f(z_0)=1$ and $\Sigma_{\nu=0}^{\infty} A_\nu\phi_\nu(z_0)=0$. Therefore by Parseval's equality (1.41) we have

$$\|f\|_D^2=\frac{\sum_{\nu=0}^{\infty}|\overline{\varphi_\nu^0}+A_\nu|^2}{K_D^2(z_0,\bar{z}_0)}=\frac{\sum_{\nu=0}^{\infty}|\varphi_\nu^0|^2+\sum_{\nu=0}^{\infty}|A_\nu|^2}{K_D^2(z_0,\bar{z}_0)}$$
$$=[K(z_0,\bar{z}_0)]^{-1}+[K(z_0,\bar{z}_0)]^{-2}\sum_{\nu=0}^{\infty}|A_\nu|^2.$$

Evidently we obtain the minimum value of $\|f\|_D^2$ if we choose all numbers A_ν equal to zero. From this our assertion follows.

REMARK. It is easy to see that the extremal property of the kernel function $K_D(z,\bar{z})$, expressed by Theorem 4.6, is equivalent to the relation

$$K_D(z,\bar{z})=\max|f(z)|^2 \tag{1.49}$$

for functions $f(z)\in L^2(D)$ satisfying the condition $\|f\|_D\leq 1$.

COROLLARY. As is clear from the proof of Theorem 4.6, *there is one and only one function* $f(z)\in L^2(D)$ *normalized by the condition* $f(z_0)=1$ *that minimizes the integral* $\|f\|_D^2$. It is given by the equation (1.48) with all $A_\nu=0$, i.e., by the formula

$$f(z)=M_D(z,z_0)=\frac{K_D(z,\bar{z}_0)}{K_D(z_0,\bar{z}_0)}. \tag{1.50}$$

Hence it turns out that the function $K_D(z,\bar{z}_0)=M_D(z,z_0)K_D(z_0,\bar{z}_0)$ is also defined by the domain D.

DEFINITION (kernel function and minimal function of a domain). The kernel function $K_D(z,\bar{\zeta})$ of all closed orthonormal systems in a bounded domain D is called the kernel function of the domain D.

The function $M_D(z,z_0)$ which minimizes the integral $\|f\|_D^2$ (equal to $[K(z_0,\bar{z}_0)]^{-1}$) for functions $f(z)\in L^2(D)$ that are normalized by the condition $f(z_0)=1$ is called the minimal function of the domain D at the point z_0.

The following theorem expresses the so-called *reproducing property* of the

kernel function of the domain.

THEOREM 4.7. *If D is a bounded domain of the space C^n_z, and a function $f(z) \in L^2(D)$, then*

$$f(z) = \int_{D_\zeta} f(\zeta) K(z, \bar{\zeta})\, dv_\zeta.$$

Here dv_ζ is the volume element in the space C^n_ζ.

PROOF. By the complement to Theorem 4.4 we have

$$\int_{D_\zeta} f(\zeta) K(z, \bar{\zeta})\, dv_\zeta = \left(\sum_{\nu=0}^{\infty} a_\nu \varphi_\nu(\zeta),\ \sum_{\nu=0}^{\infty} \overline{\varphi_\nu(z)} \varphi_\nu(\zeta) \right)_{D_\zeta}$$

$$= \sum_{\nu=0}^{\infty} a_\nu \varphi_\nu(z) = f(z).$$

Here $\{\phi_\nu,\ \nu = 0, 1, 2, \cdots\}$ is a closed orthonormal system of functions in the domain D, and a_ν are the Fourier coefficients of the function f with respect to this orthonormal system.

We shall discuss the geometrical meaning of the kernel function and the minimum function of the domain in the case $n = 1$. The integral $\|f\|^2_D$ in this case is the area of the (generally speaking, multiple-sheeted) domain arising from the domain D under the mapping by means of the function $w = \int_{z_0}^{z} f(\zeta)d\zeta$ $(z_0, z \in D)$. Thus among functions $w = F(z)$ mapping the domain D onto a domain of finite area and satisfying the condition $F'(z_0) = 1$, the function $w = \int_{z_0}^{z} M(\zeta, z_0)d\zeta$ maps this domain onto the domain of smallest area. The minimum function of the domain turns out to be the derivative of the function generating this extremal mapping (in the sense of the area of the domain obtained). This smallest area is equal to $[K_D(z_0, \bar{z}_0)]^{-1}$, namely to the inverse of the kernel function of the domain D at the point z_0.

If the domain D is single-sheeted then, as is shown in the theory of conformal mappings, the function $w = \int_{z_0}^{z} M(\zeta, z_0)\, d\zeta$ maps this domain onto a disk with its center at the point $w = 0$ $(z = z_0$ is carried into this point). Since the area of this disk is equal to $[K_D(z_0, \bar{z}_0)]^{-1}$, its radius is equal to $[\pi K_D(z_0, \bar{z}_0)]^{-1/2}$. In the theory of conformal mappings one usually considers the function $w = w(z)$ which realizes a biholomorphic mapping of the single-sheeted domain D onto the disk $|w| < 1$ and is subject to the conditions $w(z_0) = 0$ and $w'(z_0) > 0$. From the above argument it follows that in this case $w'(z_0) = [\pi K_D(z_0, \bar{z}_0)]^{1/2}$.

The significance of the kernel function and the minimum function of the domain

in the general theory of biholomorphic mappings will become clear below.

In closing this subsection we derive a theorem which is useful for finding the kernel function of the product of domains.

THEOREM 4.8 (Bremermann [2], (I)). *If $B \subset C^m_w$ and $D \subset C^n_z$ are bounded domains, in the spaces of the variables $w_1, \cdots, w_m$ and $z_1, \cdots, z_n$ respectively, and the domain $G = B \times D \subset C^{n+m}_{w,z}$, then*

$$K_G(w, z, \bar{\omega}, \bar{\zeta}) = K_B(w, \bar{\omega}) K_D(z, \bar{\zeta}).$$

Here $K_G(w, z, \bar{\omega}, \bar{\zeta}) = K_G(w_1, \cdots, w_m, z_1, \cdots, z_n, \bar{\omega}_1, \cdots, \bar{\omega}_m, \bar{\zeta}_1, \cdots, \bar{\zeta}_n)$ and the points $w \in B_w$, $z \in D_z$, $\omega \in B_\omega$, $\zeta \in D_\zeta$.

PROOF. Let a function $f(w, z) \in L^2(G)$. Then the functions $f(w, z^{(0)})$ and $f(w^{(0)}, z)$ are holomorphic respectively in the domains B and D. Here $z^{(0)}$ is a fixed point of the domain D and $w^{(0)}$ is a fixed point of the domain B. We shall show that $f(w, z^{(0)}) \in L^2(B)$.

In view of the lemma to Theorem 4.1, if the points $w \in B$, $z^{(0)} \in D$ we have

$$|f(w, z^{(0)})|^2 \leqslant \frac{1}{\pi^n r^{2n}} \int_E |f(w, z)|^2 dv_z.$$

Here the integral is taken over the polycylinder $E = \{|z_j - z_j^{(0)}| < r,\ j = 1, \cdots, n\}$. Hence it follows that

$$\int_B |f(w, z^{(0)})|^2 dv_w \leqslant \frac{1}{\pi^n r^{2n}} \int_B \left[\int_E |f(w, z)|^2 dv_z\right] dv_w$$
$$\leqslant \frac{1}{\pi^n r^{2n}} \int_G |f(w, z)|^2 dv_{w,z} < \infty.$$

From this inequality our assertion then follows. In the same manner we can also establish that $f(w^{(0)}, z) \in L^2(D)$.

Let $\{\phi_\nu(w),\ \nu = 0, 1, 2, \cdots\}$ and $\{\psi_\nu(z),\ \nu = 0, 1, 2, \cdots\}$ be closed orthonormal systems in the domains B and D, respectively. Then, so long as $f(w, z^{(0)}) \in L^2(B)$ for $w \in B$ and $z^{(0)} \in D$, we have

$$f(w, z^{(0)}) = \sum_{\nu=0}^{\infty} a_\nu(z^{(0)}) \varphi_\nu(w), \tag{1.51}$$

where $a_\nu(z) = \int_B f(w, z) \overline{\phi_\nu(w)}\, dv_w$ is a holomorphic function of the variables $z_1, \cdots, z_n$ in the domain D. This last fact may be shown by verifying the Cauchy-Riemann conditions for the functions $a_\nu(z)$. To this end we must use the

holomorphicity of the function $f(w, z)$ and the usual rules for differentiation of an integral with respect to a parameter.

From the expansion (1.51) it follows that

$$\int_B |f(w, z)|^2 dv_w = \sum_{\nu=0}^{\infty} |a_\nu(z)|^2 \geqslant |a_{\nu_0}(z)|^2, \tag{1.52}$$

where ν_0 is an arbitrary natural number. Integrating both sides of the inequality (1.52) over the domain D, we further obtain

$$\infty > \int_G |f(w, z)|^2 dv_{w,z} \geqslant \int_D |a_{\nu_0}(z)|^2 dv_z.$$

Thus we have established that $a_\nu(z) \in L^2(D)$ for all numbers ν.

Therefore in the domain D the representations

$$a_\nu(z) = \sum_{\mu=0}^{\infty} a_{\mu\nu}\psi_\mu(z)$$

and thus

$$f(w, z) = \sum_{\nu=0}^{\infty} \left(\sum_{\mu=0}^{\infty} a_{\mu\nu}\psi_\mu(z) \right) \varphi_\nu(w) \tag{1.53}$$

hold.

Applying the Bunjakovskiĭ-Schwarz inequality to the series (1.53) we establish that

$$\left| \sum_{\nu=0}^{\infty} \left(\sum_{\mu=0}^{\infty} a_{\mu\nu}\psi_\mu(z) \right) \varphi_\nu(w) \right|^2$$
$$\leqslant \sum_{\nu=0}^{\infty} \left| \sum_{\mu=0}^{\infty} a_{\mu\nu}\psi_\mu(z) \right|^2 \sum_{\nu=0}^{\infty} |\varphi_\nu(w)|^2$$
$$\leqslant \sum_{\nu=0}^{\infty} \left(\sum_{\mu=0}^{\infty} |a_{\mu\nu}|^2 \cdot \sum_{\mu=0}^{\infty} |\psi_\mu(z)|^2 \right) \sum_{\nu=0}^{\infty} |\varphi_\nu(w)|^2$$
$$= K_B(w, \bar{w}) K_D(z, \bar{z}) \sum_{\nu, \mu=0}^{\infty} |a_{\mu\nu}|^2.$$

On the other hand it is easy to see that $\Sigma_{\mu,\nu=0}^{\infty} |a_{\mu\nu}| < \infty$. Therefore the series

$$f(w, z) = \sum_{\mu, \nu=0}^{\infty} a_{\mu\nu}\psi_\mu(z) \varphi_\nu(w)$$

converges absolutely and uniformly in the domain G. By a direct calculation it turns out that the functions $\phi_\nu(w)\psi_\mu(z)$, where $\nu, \mu = 0, 1, 2, \cdots$, form an orthonormal system in the domain G.

Now, on the basis of Theorem 4.4, we conclude that this system is closed. But then

$$K_G(w, z, \bar{\omega}, \bar{\zeta}) = \sum_{\mu, \nu=0}^{\infty} \varphi_\nu(w)\psi_\mu(z)\overline{\varphi_\nu(\omega)}\,\overline{\psi_\mu(\zeta)}$$

$$= \left(\sum_{\nu=0}^{\infty} \varphi_\nu(w)\overline{\varphi_\nu(\omega)}\right)\left(\sum_{\mu=0}^{\infty} \psi_\mu(z)\overline{\psi_\mu(\zeta)}\right) = K_B(w, \bar{\omega})K_D(z, \bar{\zeta}).$$

Thus Theorem 4.8 is proved.

4. Examples. Consider an n-circular domain $D = \{|z_n| < g(|z_1|, \cdots, |z_{n-1}|), |z_j| < R_j, \ j = 1, \cdots, n-1\}$. Here $g(r_1, \cdots, r_{n-1})$ is a bounded function integrable in the sense of Lebesgue for $0 \le r_j \le R_j$, $j = 1, \cdots, n-1$. We consider the monomial

$$\varphi_\nu = \frac{1}{a_{p_1 \cdots p_n}} z_1^{p_1} \ldots z_n^{p_n},$$

where $\nu = (p_1 \cdots p_n)$ and

$$a^2_{p_1 \cdots p_n} = \int_D z_1^{p_1} \ldots z_n^{p_n} \bar{z}_1^{p_1} \ldots \bar{z}_n^{p_n} dv.$$

The functions $\{\phi_\nu, \ \nu = 0, 1, 2, \cdots\}$ form a closed orthonormal system. Its orthonormality follows from the fact that (here $z_j = r_j e^{i\theta_j}$, $j = 1, \cdots, n$; $\nu = (p_1 \cdots p_n)$, $\nu' = (p_1' \cdots p_n')$)

$$\int_D z_1^{p_1} \ldots z_n^{p_n} \bar{z}_1^{p_1'} \ldots \bar{z}_n^{p_n'} dv$$

$$= \int_0^{R_1} dr_1 \ldots \int_0^{R_{n-1}} dr_{n-1} \int_0^{g(r_1, \ldots, r_{n-1})} dr_n \int_0^{2\pi} d\varphi_1 \ldots$$

$$\ldots \int_0^{2\pi} d\varphi_{n-1} \int_0^{2\pi} r_1^{p_1+p_1'+1} \ldots r_n^{p_n+p_n'+1} \exp\left[\sum_{j=1}^{n} i\varphi_j(p_j - p_j')\right] d\varphi_n$$

$$= \begin{cases} 0 & \text{for} \quad \nu \neq \nu', \\ a^2_{p_1 \cdots p_n} & \text{for} \quad \nu = \nu'. \end{cases}$$

The closure of the function system $\{\phi_\nu, \nu = 0, 1, 2, \cdots\}$ follows, by Theorem 4.4, from the fact that every function holomorphic in the n-circular domain D can be represented there by a power series $\sum_k c_k z^k = \sum_{\nu=0}^{\infty} \alpha_\nu \phi_\nu$.

Special cases. a) *Hyperball* $B = \{|z_1|^2 + \cdots + |z_n|^2 < R^2\}$. Calculation shows that in this case

$$a^2_{p_1 \cdots p_n} = \frac{p_1! \ldots p_n!}{(p_1 + \ldots + p_n + n)!} \pi^n R^{2(p_1 + \cdots + p_n + n)}.$$

Consequently, if points $z, \zeta \in B$, we have

$$\begin{aligned} K_E(z, \bar{\zeta}) &= \sum_{p_1, \ldots, p_n = 0}^{\infty} \frac{1}{a^2_{p_1 \cdots p_n}} z_1^{p_1} \ldots z_n^{p_n} \bar{\zeta}_1^{p_1} \ldots \bar{\zeta}_n^{p_n} \\ &= \frac{1}{\pi^n} \sum_{\alpha=0}^{\infty} \frac{(\alpha+1)\ldots(\alpha+n)}{R^{2(\alpha+n)}} \sum_{p_1 + \cdots + p_n = \alpha} \frac{\alpha!}{p_1! \ldots p_n!} (z_1\bar{\zeta}_1)^{p_1} \ldots \\ &\ldots (z_n\bar{\zeta}_n)^{p_n} = \frac{1}{\pi^n} \sum_{\alpha=0}^{\infty} \frac{(\alpha+1)\ldots(\alpha+n)}{R^{2(\alpha+n)}} (z_1\bar{\zeta}_1 + \ldots + z_n\bar{\zeta}_n)^{\alpha} \\ &= \frac{1}{(\pi R^2)^n} \sum_{\alpha=0}^{\infty} (\alpha+1)\ldots(\alpha+n) \left(\frac{z_1\bar{\zeta}_1 + \ldots + z_n\bar{\zeta}_n}{R^2} \right)^{\alpha}. \end{aligned}$$

By the use of the identity

$$\sum_{\alpha=0}^{\infty} (\alpha+1)\ldots(\alpha+n) q^{\alpha} = \frac{d^n}{dq^n}\left(\frac{1}{1-q}\right) = \frac{n!}{(1-q)^{n+1}} \quad \text{for} \quad |q|<1,$$

and the fact that $|z_1\bar{\zeta}_1 + \cdots + z_n\bar{\zeta}_n| < R^2$ we obtain

$$K_B(z, \bar{\zeta}) = \frac{n!\, R^2}{\pi^n \left(R^2 - \sum_{k=1}^{n} z_k \bar{\zeta}_k \right)^{n+1}}. \tag{1.54}$$

b) *Polycylinder* $E = \{|z_j| < R_j, \ j = 1, \cdots, n\}$. Using formula (1.54) and Theorem 4.8 we find that

$$K_E(z, \bar{\zeta}) = \frac{R_1^2 \ldots R_n^2}{\pi^n (R_1 - z_1\bar{\zeta}_1)^2 \ldots (R_n - z_n\bar{\zeta}_n)^2}. \tag{1.55}$$

c) *Bicircular domain* $D = \{\alpha|w|^{2/p} + |z^2| < 1\}$ *in the space* C^2 *of the variables* w *and* z. Here p is a natural number and $0 < \alpha \leq 1$. By calculation we find that

$$a_{mn}^2 = \frac{\pi^2 p\,[(m+1)p-1]!\,n!}{\alpha^{p(m+1)}\,[(m+1)p+n+1]!},$$

and

$$K_D(w,\, z,\, \bar{\omega},\, \bar{\zeta})$$
$$= \frac{\alpha^p (1-z\bar{\zeta})^{p-2}\,[(p+1)(1-z\bar{\zeta})^p + (p-1)\,\alpha^p w\bar{\omega}]}{\pi^2\,[(1-z\bar{\zeta})^p - \alpha^p w\bar{\omega}]^3}. \tag{1.56}$$

For $\alpha = p = 1$ the formula (1.56) is reduced to the formula (1.54) for the case $n = 2$ and $R = 1$.

d) *Disk domain* $G = \{|w|^2 + |z|^2 < \phi^2(s)\}$, *where* $s = wz^{-1}$. The boundary of this domain is given by the equation $|z| = \phi(s)\,(1+s^2)^{-1/2} = R(s)$; we again assume the function $R(s)$ to be integrable in the sense of Lebesgue.

For the construction of a closed system of orthonormal functions we shall use the fact that the monomials $w^n z^{m-n}$ and $w^{n_1} z^{m_1 - n_1}$, for $m \neq m_1$, are orthogonal to each other in the domain G. In fact we have

$$\int_{D_0} w^n z^{m-n}\,\overline{w^{n_1} z^{m_1-n_1}}\, dv$$
$$= \frac{1}{4} \int_{|s|<\infty} s^n \bar{s}^{n_1}\, d\bar{s}\, ds \int_{\epsilon<|z|<R(s)} z^{m+1}\,\bar{z}^{m_1+1}\, d\bar{z}\, dz = 0,$$

since z^{m+1} and z^{m_1+1} $(m \neq m_1)$ are orthogonal in the annulus $\epsilon < |z| < R(s)$, where $D_0 = \{\epsilon < |z| < R(s),\ |s| < \infty\}$. We set

$$dv = \left| -\frac{1}{4}\, d\bar{w} \wedge dw \wedge d\bar{z} \wedge dz \right|$$
$$= \left| -\frac{1}{4}\, \frac{\partial(\bar{w},\, w,\, \bar{z},\, z)}{\partial(\bar{s},\, s,\, \bar{z},\, z)}\, d\bar{s} \wedge ds \wedge d\bar{z} \wedge dz \right|$$
$$= \left| -\frac{1}{4}\, \bar{z} z\, d\bar{s} \wedge ds \wedge d\bar{z} \wedge dz \right|.$$

Now, letting ϵ go to zero we obtain $(w^n z^{m-n},\ w^{n_1} z^{m_1 - n_1})_G = 0$ for $m \neq m_1$. Therefore it only remains for us to orthogonalize and to normalize the functions $z^m,\ wz^{m-1},\ \cdots,\ w^n$ for each m. This is easily carried out by Schmidt's well-known process as follows: take, for example, $c_{11} z^m$ as the first function, $c_{12} z^m + c_{22} w z^{m-1}$ as the second, $c_{13} z^m + c_{23} w z^{m-1} + c_{33} w^2 z^{m-2}$ as the third, and so forth. The coefficients c_{ik} are so chosen that the succeeding functions are normalized and orthogonal with respect to the preceding ones. Thus we obtain an orthonormal function system consisting of homogeneous polynomials. It can be shown that the function system thus obtained is closed.

e) *Semidisk domain* $H = \{|z| < R(w),\ |w| < 1\}$. Reasoning in the same way as in the preceding cases, we can show that there will exist a closed orthonormal

system of functions $\phi_{mn}(w, z) = z^m f_{mn}(w)$. Here $f_{mn}(w)$ are functions holomorphic in the disk $|w| < 1$.

In §24 of Chapter V we will derive the value of the kernel function of a wide class of so-called homogeneous domains (see Theorem 24.8).

5. Some variational problems. The variational problem whose solution turns out to be the inverse of the kernel function of a domain is a special case of a problem of more general type. The solution of the latter problem is useful for the theory of biholomorphic mappings.

We shall seek the minimum of the integral $I = \|f\|_D$. We impose on the functions $f \in L^2(D)$ the following conditions:

Let $f(w, z) = \sum_{k=1}^{\infty} A_k \phi_k(w, z)$; it is required that the coefficients A_k satisfy the equations

$$\sum_{s=1}^{\infty} A_s \alpha_{qs} = x_q; \qquad q = 1, \ldots, n. \tag{1.57}$$

Here x_q, α_{qs} are given constants, while the numbers α_{qs} are to be chosen so that the series $\sum_{s=1}^{\infty} |\alpha_{qs}|^2$ converges for $q = 1, \cdots, n$.

To find the minimum of $I = \|f\|_D = \sum_{s=1}^{\infty} |A_s|^2$ under the above-mentioned auxiliary conditions, it is always more convenient to carry out the computation by Lagrange's method. As a result of calculation we obtain (setting $\sum_{s=1}^{\infty} \alpha_{ps}\bar{\alpha}_{qs} = c_{p\bar{q}}$):

$$I_{\min} = -\frac{\begin{vmatrix} 0 & \bar{x}_1 & \bar{x}_2 & \ldots & \bar{x}_n \\ x_1 & c_{1\bar{1}} & c_{1\bar{2}} & \cdots & c_{1\bar{n}} \\ x_2 & c_{2\bar{1}} & c_{2\bar{2}} & \cdots & c_{2\bar{n}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_n & c_{n\bar{1}} & c_{n\bar{2}} & \ldots & c_{n\bar{n}} \end{vmatrix}}{\begin{vmatrix} c_{1\bar{1}} & c_{1\bar{2}} & \cdots & c_{1\bar{n}} \\ c_{2\bar{1}} & c_{2\bar{2}} & \cdots & c_{2\bar{n}} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n\bar{1}} & c_{n\bar{2}} & \cdots & c_{n\bar{n}} \end{vmatrix}}$$

$$= \sum_{k=1}^{n} \frac{\left\|\begin{matrix} x_1 & c_{1\bar{1}} & \cdots & c_{1,\overline{k-1}} \\ x_2 & c_{2\bar{1}} & \cdots & c_{2,\overline{k-1}} \\ \cdot & \cdot & \cdot & \cdot \\ x_k & c_{k\bar{1}} & \cdots & c_{k,\overline{k-1}} \end{matrix}\right\|^2}{\begin{vmatrix} c_{1\bar{1}} & c_{1\bar{2}} & \cdots & c_{1,\overline{k-1}} \\ c_{2\bar{1}} & c_{2\bar{2}} & \cdots & c_{2,\overline{k-1}} \\ \cdot & \cdot & \cdot & \cdot \\ c_{k-1,\bar{1}} & c_{k-1,\bar{2}} & \cdots & c_{k-1,\overline{k-1}} \end{vmatrix} \cdot \begin{vmatrix} c_{1\bar{1}} & c_{1\bar{2}} & \cdots & c_{1\bar{k}} \\ c_{2\bar{1}} & c_{2\bar{2}} & \cdots & c_{2\bar{k}} \\ \cdot & \cdot & \cdot & \cdot \\ c_{k\bar{1}} & c_{k\bar{2}} & \cdots & c_{k\bar{k}} \end{vmatrix}} \tag{1.58}$$

Examples. 1) The conditions (1.57) consist of $f(a, b) = x_{00}$; $(\partial f/\partial w)_{a,b} = x_{10}$; $(\partial f/\partial z)_{a,b} = x_{01}$. In this case $\alpha_{1s} = \phi_s(a, b)$, $\alpha_{2s} = (\partial\phi_s/\partial w)_{a,b}$, $\alpha_{3s} = (\partial\phi_s/\partial z)_{a,b}$. Using equation (1.37) and setting (here and in what follows)

$$\left(\frac{\partial^{m+n+p+q}K}{\partial w^m\,\partial z^n\,\partial\bar{w}^p\,\partial\bar{z}^q}\right)_{a,\,b} = K_{mn\bar{p}\bar{q}},$$

we find, starting from (1.58) (we shall denote the minimum by $I_1(x_{00}, x_{10}, x_{01})$):

$$I_1(x_{00}, x_{10}, x_{01}) = \frac{1}{K}\,|\,x_{00}\,|^2 + \frac{\left\|\begin{matrix} x_{00} & K \\ x_{10} & K_{10\bar{0}\bar{0}} \end{matrix}\right\|^2}{K\begin{vmatrix} K & K_{00\bar{1}\bar{0}} \\ K_{10\bar{0}\bar{0}} & K_{10\bar{1}\bar{0}} \end{vmatrix}} + \frac{\left\|\begin{matrix} x_{00} & K & K_{00\bar{1}\bar{0}} \\ x_{10} & K_{10\bar{0}\bar{0}} & K_{10\bar{1}\bar{0}} \\ x_{01} & K_{01\bar{0}\bar{0}} & K_{01\bar{1}\bar{0}} \end{matrix}\right\|^2}{\begin{vmatrix} K & K_{00\bar{1}\bar{0}} \\ K_{10\bar{0}\bar{0}} & K_{10\bar{1}\bar{0}} \end{vmatrix}\begin{vmatrix} K & K_{00\bar{1}\bar{0}} & K_{00\bar{0}\bar{1}} \\ K_{10\bar{0}\bar{0}} & K_{10\bar{1}\bar{0}} & K_{10\bar{0}\bar{1}} \\ K_{01\bar{0}\bar{0}} & K_{01\bar{1}\bar{0}} & K_{01\bar{0}\bar{1}} \end{vmatrix}}. \tag{1.59}$$

We agree that the function minimizing the integral I to the value I_1 will be denoted by $f_{(x_{00}, x_{10}, x_{01})}(w, z)$. The expression for it could be constructed on the basis of equation (1.59).

2) The conditions (1.57) have the form

$$f(a, b) = 0,\ p\left(\frac{\partial f}{\partial w}\right)_{a,\,b} + q\left(\frac{\partial f}{\partial z}\right)_{a,\,b} = 1.$$

In this case

$$\alpha_{1s} = \varphi_s(a, b),\ \alpha_{2s} = p\left(\frac{\partial\varphi_s}{\partial w}\right)_{a,\,b} + q\left(\frac{\partial\varphi_s}{\partial z}\right)_{z,\,b}.$$

Again using equation (1.37) and starting from formula (1.58) we obtain (the minimum is denoted by I_2):

$$I_2 = \frac{K}{\begin{vmatrix} K & \bar{p}K_{00\bar{1}\bar{0}} + \bar{q}K_{00\bar{0}\bar{1}} \\ pK_{10\bar{0}\bar{0}} + qK_{01\bar{0}\bar{0}}; & p\bar{p}K_{10\bar{1}\bar{0}} + p\bar{q}K_{10\bar{0}\bar{1}} + q\bar{p}K_{01\bar{1}\bar{0}} + q\bar{q}K_{01\bar{0}\bar{1}} \end{vmatrix}}. \tag{1.60}$$

3) The conditions (1.57) have the form $f(a, b) = (\partial f/\partial w)_{a,b} = (\partial f/\partial z)_{a,b} = 0$, $p^2(\partial^2 f/\partial w^2)_{a,b} + 2pq(\partial^2 f/\partial w\partial z)_{a,b} + q^2(\partial^2 f/\partial z^2)_{a,b} = 1$. In this case $\alpha_{1s} = \phi_s(a, b)$, $\alpha_{2s} = (\partial\phi_s/\partial w)_{a,b}$, $\alpha_{3s} = (\partial\phi_s/\partial z)_{a,b}$, $\alpha_{4s} = p^2(\partial^2\phi_s/\partial w^2)_{a,b} + 2pq(\partial^2\phi_s/\partial w\partial z)_{a,b} + q^2(\partial^2\phi_s/\partial z^2)_{a,b}$. Starting from equation (1.37) we obtain (the minimum is denoted by I_3):

$$I_3 = \frac{\begin{vmatrix} K & K_{00\bar{1}\bar{0}} & K_{00\bar{0}\bar{1}} \\ K_{10\bar{0}\bar{0}} & K_{10\bar{1}\bar{0}} & K_{10\bar{0}\bar{1}} \\ K_{01\bar{0}\bar{0}} & K_{01\bar{1}\bar{0}} & K_{01\bar{0}\bar{1}} \end{vmatrix}}{\begin{vmatrix} K & K_{00\bar{1}\bar{0}} & K_{00\bar{0}\bar{1}} & \bar{A}_{\bar{0}\bar{0}} \\ K_{10\bar{0}\bar{0}} & K_{10\bar{1}\bar{0}} & K_{10\bar{0}\bar{1}} & \bar{A}_{\bar{1}\bar{0}} \\ K_{01\bar{0}\bar{0}} & K_{01\bar{1}\bar{0}} & K_{01\bar{0}\bar{1}} & \bar{A}_{\bar{0}\bar{1}} \\ A_{\bar{0}\bar{0}} & A_{\bar{1}\bar{0}} & A_{\bar{0}\bar{1}} & H \end{vmatrix}}. \tag{1.61}$$

Here

$$A_{\bar{m}\bar{n}} = p^2 K_{20\bar{m}\bar{n}} + 2pqK_{11\bar{m}\bar{n}} + q^2 K_{02\bar{m}\bar{n}};$$

$$\begin{aligned} H = {} & p^2\bar{p}^2 K_{20\bar{2}\bar{0}} + 2p^2\overline{pq}K_{20\bar{1}\bar{1}} + p^2\bar{q}^2 K_{20\bar{0}\bar{2}} + 2pq\bar{p}^2 K_{11\bar{2}\bar{0}} \\ & + 4pq\overline{pq}K_{11\bar{1}\bar{1}} + 2pq\bar{q}^2 K_{11\bar{0}\bar{2}} + q^2\bar{p}^2 K_{02\bar{2}\bar{0}} \\ & + 2q^2\overline{pq}K_{02\bar{1}\bar{1}} + q^2\bar{q}^2 K_{02\bar{0}\bar{2}}. \end{aligned}$$

In the last two minimal problems p and q are certain constants.

Analogously, one can also form the function $f(w, z)$ which minimizes the integral I to the minimum value (1.58); it is of the form

$$f(w, z) = -\frac{\begin{vmatrix} 0 & F_1 & F_2 & \dots & F_n \\ x_1 & c_{1\bar{1}} & c_{1\bar{2}} & \dots & c_{1\bar{n}} \\ x_2 & c_{2\bar{1}} & c_{2\bar{2}} & \dots & c_{2\bar{n}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_n & c_{n\bar{1}} & c_{n\bar{2}} & \dots & c_{n\bar{n}} \end{vmatrix}}{\begin{vmatrix} c_{1\bar{1}} & c_{1\bar{2}} & \dots & c_{1\bar{n}} \\ c_{2\bar{1}} & c_{2\bar{2}} & \dots & c_{2\bar{n}} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n\bar{1}} & c_{n\bar{2}} & \dots & c_{n\bar{n}} \end{vmatrix}} =$$

$$
= \sum_{k=1}^{n} \frac{\begin{vmatrix} F_1 & F_2 & \cdots & F_k \\ c_{1\bar{1}} & c_{1\bar{2}} & \cdots & c_{1\bar{k}} \\ \cdot & \cdot & \cdot & \cdot \\ c_{k-1,\bar{1}} & c_{k-1,\bar{2}} & \cdots & c_{k-1,\bar{k}} \end{vmatrix} \cdot \begin{vmatrix} x_1 & c_{1\bar{1}} & \cdots & c_{1,\overline{k-1}} \\ x_2 & c_{2\bar{1}} & \cdots & c_{2,\overline{k-1}} \\ \cdot & \cdot & \cdot & \cdot \\ x_k & c_{k\bar{1}} & \cdots & c_{k,\overline{k-1}} \end{vmatrix}}{\begin{vmatrix} c_{1\bar{1}} & c_{1\bar{2}} & \cdots & c_{1\bar{k}} \\ c_{2\bar{1}} & c_{2\bar{2}} & \cdots & c_{2\bar{k}} \\ \cdot & \cdot & \cdot & \cdot \\ c_{k\bar{1}} & c_{k\bar{2}} & \cdots & c_{k\bar{k}} \end{vmatrix} \cdot \begin{vmatrix} c_{1\bar{1}} & c_{1\bar{2}} & \cdots & c_{1,\overline{k-1}} \\ c_{2\bar{1}} & c_{2\bar{2}} & \cdots & c_{2,\overline{k-1}} \\ \cdot & \cdot & \cdot & \cdot \\ c_{k-1,\bar{1}} & c_{k-1,\bar{2}} & \cdots & c_{k-1,\overline{k-1}} \end{vmatrix}} . \tag{1.62}
$$

Here $F_k = \Sigma_{s=1}^{\infty} \bar{\alpha}_{ks}\phi_s(w, z)$.

We remark that, as will be seen later, by means of the quantities I_1, I_2, I_3 and other solutions of minimal problems of the type (1.58), we can express various quantities that are characteristic to the Bergman metric which is invariant under biholomorphic mappings. This circumstance ("principle of minimal problems") makes it possible to establish a series of properties of these quantities.

The following obvious proposition has an important meaning:

If $I^{(A)}_{\min}$, $I^{(B)}_{\min}$ are solutions of two minimal problems with one and the same auxiliary conditions (1.57) for domains A and B, and if the domain A lies inside the domain B, then

$$
I^{(A)}_{\min} \leq I^{(B)}_{\min}. \tag{1.63}
$$

In particular, for $A \subset B$ and $z \in A$

$$
K_B(z, \bar{z}) \leq K_A(z, \bar{z}) \tag{1.64}
$$

For $n = 1$ the inequality (1.64) expresses Schwarz's lemma in its invariant formulation (namely Pick's theorem).

§5. PROPERTIES OF THE KERNEL FUNCTION OF A DOMAIN

1. **Change of kernel function under biholomorphic mappings.** Let D be a bounded domain of the space C^n of the variables $z^1, \cdots, z^n$; functions

$$
\begin{aligned}
&(\tau) \qquad & z^{*k} &= g^k(z), \qquad k = 1, \ldots, n, \\
&(\tau^{-1}) \qquad & z^k &= h^k(z^*), \qquad k = 1, \ldots, n,
\end{aligned} \tag{1.65}
$$

define a biholomorphic mapping of this domain D onto a domain D^* of the space C^{*n} of the variables $z^{*1}, \cdots, z^{*n}$ and inversely that of the domain D^* onto the domain D.

Then the Jacobian

$$J = \frac{\partial z}{\partial z^*} = \frac{\partial (h^1, \ldots, h^n)}{\partial (z^{*1}, \ldots, z^{*n})}$$

differs from zero for $z^* \in D^*$. Let $\{\phi_\nu, \ \nu = 0, 1, 2, \cdots\}$ be a closed orthonormal system of functions in the domain D. If we make the substitution (τ^{-1}) in the integral of (1.34), then the condition (1.34) assumes the form

$$[\varphi_\nu (h(z^*)) J, \ \varphi_\mu (h(z^*)) J]_{D^*} = \begin{cases} 0 & \text{for} \quad \nu \neq \mu, \\ 1 & \text{for} \quad \nu = \mu. \end{cases}$$

Hence we see that the functions $\phi_\nu(h(z^*))J = \phi^*_\nu(z^*)$, $\nu = 0, 1, 2, \cdots$, form an orthonormal system. Every function $f^*(z^*) \in L^2(D^*)$ can be represented in a form $f^* = fJ$, where $f(z) = f^*(g(z))J^{-1} \in L^2(D)$ (as can be seen by making the change of variables in the integral (1.32)). Multiplying equation (1.44) by J we establish, on the basis of Theorem 4.4, that the system $\{\phi^*_\nu(z^*), \ \nu = 0, 1, 2, \cdots\}$ is closed in the domain D^*. We now see that

$$K_{D^*}(z^*, \overline{\zeta^*}) = \sum_{\nu=0}^{\infty} \varphi^*_\nu(z^*) \overline{\varphi^*_\nu(\zeta^*)} = \sum_{\nu=0}^{\infty} \varphi_\nu(z) \overline{\varphi_\nu(\zeta)} \left(\frac{\partial z}{\partial z^*}\right)\left(\frac{\overline{\partial \zeta}}{\partial \zeta^*}\right)$$

$$= K_D(z, \overline{\zeta}) \left(\frac{\partial z}{\partial z^*}\right)\left(\frac{\overline{\partial \zeta}}{\partial \zeta^*}\right).$$

Thus we have proved the

THEOREM 5.1 (Bergman [5]). *For the biholomorphic mapping of a bounded domain D onto a domain D^* the following relation holds:*

$$K_{D^*}(z^*, \overline{\zeta^*}) = K_D(z, \overline{\zeta}) \left(\frac{\partial z}{\partial z^*}\right)\left(\frac{\overline{\partial \zeta}}{\partial \zeta^*}\right). \tag{1.66}$$

We now consider the quantities

$$T^{(D)}_{l\overline{m}} = \frac{\partial^2 \ln K_D(z, \overline{z})}{\partial z^l \, \overline{\partial z^m}}. \tag{1.67}$$

Clearly they satisfy the condition $\overline{T}^{(D)}_{l\overline{m}} = T^{(D)}_{m\overline{l}}$ and are transformed under the biholomorphic mapping (1.65) as components of a covariant tensor of second rank. Indeed, by differentiating (1.66) for $\zeta = z$, $\zeta^* = z^*$ we have

$$T^{(D^*)}_{l\overline{m}} = \sum_{p,\,q=1}^{n} T^{(D)}_{p\overline{q}} \left(\frac{\partial z^p}{\partial z^{*l}}\right)\left(\frac{\overline{\partial z^q}}{\partial z^{*m}}\right), \tag{1.68}$$

where the quantities $T^{(D^*)}_{l\overline{m}}$ are defined for the domain D^* in precisely the same way as were $T^{(D)}_{l\overline{m}}$ for the domain D.

Thus equation (1.67) defines an hermitian covariant tensor of second rank. We consider the hermitian differential form $\Sigma_{l,m=1}^{n} T_{l\overline{m}} dz^l d\overline{z}^m$. From what has been said it is invariant under biholomorphic mappings. It is also positive definite. This follows from the fact that the quantity

$$\left(K_D \sum_{l,\, m=1}^{n} T_{l\overline{m}}^{(D)} u^l \,\overline{u}^m\right)^{-1}, \tag{1.69}$$

calculated at a certain point $a \in D$, is the minimal value of the integral $\|f\|_D^2$ for functions $f \in L^2(D)$ normalized by the conditions $f(a) = 0$ and $\Sigma_{k=1}^{n}(\partial f/\partial a^k)u^k = 1$. We note that we have considered the similar minimal problem for the case $n = 2$ in the second example of §4.5. Thus we arrive at the following result:

THEOREM 5.2 (Bergman [5]). *The positive definite hermitian differential form*

$$ds^2 = \sum_{l,\, m=1}^{n} T_{l\overline{m}}^{(D)} dz^l \, d\overline{z}^m, \tag{1.70}$$

where

$$T_{l\overline{m}}^{(D)} = \frac{\partial^2 \ln K_D(z, \overline{z})}{\partial z^l \, \partial \overline{z}^m}, \tag{1.71}$$

and K_D is the kernel function of a bounded domain D, defines in this domain an hermitian metric (or an hermitian geometry) invariant under biholomorphic mappings.

We note that in general a Riemann metric or geometry is said to be hermitian if it is given by a positive definite hermitian differential form. In the following the metric (1.70) is called the *Bergmen metric.*

For completeness we mention here two further propositions: [1)]

COROLLARY OF THE THEOREM 5.2. From the positivity of the quantity (1.69) it is evident that $\ln K_D(z, \overline{z})$ *is a plurisubharmonic function.* To this result the following proposition may be added:

THEOREM 5.3. *The kernel function $K_D(z, \overline{z})$ of a bounded domain D is a plurisubharmonic function.*

PROOF. The kernel function of the domain is defined as the sum of the series (1.37). The square modulus of a holomorphic function is a plurisubharmonic function (see Theorem 13.1). The sum of a uniformly convergent series of plurisubhar-

---

1) Theory of plurisubharmonic functions will be presented in §13, Chapter III. The reader who is not familiar with the simplest properties of these functions can postpone reading the end of §5.1 (without loss of understanding for the subsequent text) until he becomes acquainted with §13, Chapter III.

monic functions is again a plurisubharmonic function (see Theorem 13.1). From this our assertion follows.

REMARK. Theorem 5.3, as well as many other propositions of the present section, is valid for the kernel function of an arbitrary (not necessarily closed) orthonormal system of functions. We shall here confine ourselves to considering the most important case of the kernel function of the domain.

THEOREM 5.4. *If $B \subset C_w^m$ and $D \subset C_z^n$ are bounded domains in the spaces of the variables $w_1, \cdots, w_m$ and $z_1, \cdots, z_n$, respectively, and if the domain $G = B \times D \subset C_{w,z}^{n+m}$, then*

$$ds_G^2 = ds_B^2 + ds_D^2.$$

Here ds_G, ds_B, ds_D are line elements of the Bergman metrics in the corresponding spaces.

PROOF. The proof of this theorem may be reduced to the application of Theorem 4.8.

Further properties of the Bergman metric will be dealt with in §21, Chapter V.

2. Properties of the kernel function of a domain of holomorphy.

THEOREM 5.5. *If for some number $M > 0$ a set D_M is compact in a domain D which is assumed to be bounded, then all the connected components of this set D_M are domains of holomorphy.*

REMARK. Here and in the following theorems $D_M = \{z \in D,\ K_D(z, \bar{z}) < M\}$, where M is some positive number. We also note that Theorem 5.5 is included in the more general Theorem 13.6. This follows from Theorems 5.3 and 14.4.

PROOF. Let $\mathfrak{D}_M$ be one of the connected components of the set D_M, and let a point $z_0 \in \partial\mathfrak{D}_M$. Then $K_D(z_0, \bar{z}_0) = M$. We represent the kernel function $K_D(z_0, \bar{z}_0)$ by means of the series (1.37):

$$K_D(z, \bar{z}_0) = \sum_{\nu=0}^{\infty} \varphi_\nu(z)\overline{\varphi_\nu(z_0)}.$$

By the use of the Bunjakovskiĭ-Schwarz inequality we obtain

$$\begin{aligned}|K_D(z, \bar{z}_0)|^2 &\leqslant \sum_{\nu=0}^{\infty} |\varphi_\nu(z)|^2 \sum_{\nu=0}^{\infty} |\varphi_\nu(z_0)|^2 \\ &= K_D(z, \bar{z})\ K_D(z_0, \bar{z}_0) = MK_D(z, \bar{z}).\end{aligned}$$

Therefore, if the point $z \in D_M$, then $K_D(z, \bar{z}) < M$ and accordingly $|K_D(z, \bar{z}_0)|^2 < M^2$. Thus the analytic surface $K_D(z, \bar{z}_0) - M = 0$ lies entirely outside the domain $\mathfrak{D}_M$ and passes through the point z_0. Consequently the function $[K_D(z, \bar{z}_0) - M]^{-1}$ is holomorphic at $z \in \mathfrak{D}_M$ and the point $z = z_0 \in \partial\mathfrak{D}_M$ is a singular point of it. Such a function can be constructed at any point $z_0 \in \partial\mathfrak{D}_M$. Hence, in view of Theorem 13.6, (I), it follows that $\mathfrak{D}_M$ is a domain of holomorphy. The theorem is proved.

THEOREM 5.6. *Let D be a bounded domain of the space C^n. If for each number $M > 0$ a set D_M is compact with respect to the domain D, then* 1) *D is a domain of holomorphy;* 2) *$D_{M_1} \Subset D_{M_2}$ for $M_1 < M_2$ and $\lim_{M\to\infty} D_M = D$;* 3) *one can choose connected components $\mathfrak{D}_M$ of the sets D_M such that $\mathfrak{D}_M \Subset \mathfrak{D}_{M+1} \Subset D$ and $\lim_{M\to\infty}\mathfrak{D}_M = D$.*

PROOF. 1) Indeed, if D is not a domain of holomorphy, then there exists a point $z_0 \in (\partial D \cap H(D))$. In the neighborhood of that point the kernel function $K_D(z, \bar{z})$ is bounded (see relation (1.78) below). Then for sufficiently large values of M the set D_M could not be compact with respect to the domain D, and this contradicts our assumption.

2) For every closed set $B \subset D$ one can find a number μ such that $B \subset D_M$ for $M > \mu$ (it suffices for this to take $\mu = \max_{z\in B} K_D(z, \bar{z})$). This situation is also expressed by the relation $\lim_{M\to\infty} D_M = D$. The inequality $D_{M_1} \Subset D_{M_2}$ for $M_1 < M_2$ results immediately from the definition of the set D_M itself.

3) For some number $m > 0$ we take a certain component of the set D_m and denote it by $\mathfrak{D}_m$; then for the number $m + 1$ we take that connected component $\mathfrak{D}_{m+1}$ of the set D_{m+1} which contains the domain $\mathfrak{D}_m$, and so forth; as a result we form a sequence of domains of holomorphy $\mathfrak{D}_M$, $M = m, m+1, \cdots$, satisfying the condition $\mathfrak{D}_M \Subset \mathfrak{D}_{M+1} \Subset D$. We shall show that $\lim_{M\to\infty}\mathfrak{D}_M = D$.

Suppose again that the closed set $B \subset D$ and that $\mu = \max_{z\in B} K_D(z, \bar{z})$. Then $B \subset D_M$ for $M \geq \mu + 1$. If the set B is not contained in the connected component $\mathfrak{D}_{M'}$ of the set $D_{M'}$ for some value $M' \geq \mu + 1$, we connect a point $z' \in \mathfrak{D}_{M'}$ with one of the points $z'' \in B$ by a line $S \subset D$. Let $\mu_1 = \max_{z\in S} K_D(z, \bar{z})$. Then for $M \geq \max(\mu, \mu_1)$ the line S and the domain $\mathfrak{D}_M$ must be contained in the one and the same connected component of the set D_M, which evidently coincides with the domain $\mathfrak{D}_M$. Thus $B \subset \mathfrak{D}_M$ and accordingly $\lim_{M\to\infty}\mathfrak{D}_M = D$.

If the assumptions of Theorem 5.6 are satisfied for the domain D, then one usually says that *the kernel function $K_D(z, \bar{z})$ becomes indefinitely large as the*

point $z \in D$ *approaches the boundary of the domain* D.

THEOREM 5.7. *The kernel function* $K_\Delta(z, \bar{z})$ *of a Weil bounded polyhedron* Δ *becomes indefinitely large on the approach of the point* $z \in \Delta$ *to the boundary* $\partial\Delta$ *of this polyhedron.*

PROOF. By definition (see §22.3, Chapter IV, (I)) the polyhedron is one of the connected components of the set $\{z \in G, |f_j(z)| < 1, j = 1, \cdots, N\}$, where $f_j(z)$ are functions holomorphic in some domain $G \Supset \Delta$.

We shall show that $D_M \Subset \Delta$ for any number $M > 0$.

Put $\epsilon = 1/M$. So long as the polyhedron Δ is bounded, we can choose a number a such that

$$\int_{\Delta-\Delta_a} dv < \frac{\varepsilon}{4}.$$

Here $0 < a < 1$ and $\Delta_a = \{z \in \Delta, |f_j(z)| < a, j = 1, \cdots, N\}$. By the use of the fact that $|f_j(z)| < a < 1$ for $z \in \Delta_a$, $j = 1, \cdots, N$, we further choose a number p such that

$$\int_{\Delta_a} |f_j(z)|^{2p}\, dv < \frac{\varepsilon}{4}, \qquad j = 1, \ldots, N.$$

Then for all $j = 1, \cdots, N$ we have

$$\int_{\Delta} |f_j(z)|^{2p}\, dv = \int_{\Delta_a} |f_j(z)|^{2p}\, dv + \int_{\Delta-\Delta_a} |f_j(z)|^{2p}\, dv$$
$$< \int_{\Delta_a} |f_j(z)|^{2p}\, dv + \int_{\Delta-\Delta_a} dv < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

We now consider the set Δ_m with $m = 2^{-2p}$ and an arbitrary point $z^{(0)} \in \Delta \setminus \Delta_m$. Then among the functions $f_j(z)$ there is at least one function $f_{j0}(z)$ for which $|f_{j0}(z^{(0)})|^{2p} \geq 1/2$. Put $g(z) = \sqrt{2}\,[f_{j0}(z)]^p$. Then $|g(z^{(0)})| \geq 1$ and

$$\int_{\Delta} |g(z)|^2 dv = 2 \int_{\Delta} |f_{j_0}(z)|^2\, dv < \varepsilon.$$

Hence it follows that the minimum of the integral $\|h\|_\Delta^2$ for functions $h \in L^2(\Delta)$ satisfying the condition $|h(z^{(0)})| \geq 1$ (as we have seen, the function $g(z)$ belongs to these functions) is equal to $[K_\Delta(z_0, \bar{z}_0)]^{-1}$ and is less than the number ϵ. Consequently at any point $z^{(0)} \in \Delta \setminus \Delta_m$

$$K_\Delta(z^{(0)}, \bar{z}^{(0)}) > M$$

and thus $D_M \subset \Delta_m \Subset \Delta$. With this our assertion is proved.

REMARK. In the course of the proof of Theorem 5.7 we have not used all the properties stated in the definition of the Weil analytic polyhedron (see §22.1, Chapter IV, (I)) and the assertion of the theorem turns out to be valid for a wider class of domains. We formulate this theorem for Weil polyhedra which are the most important domains of this type.

From Theorems 5.6 and 5.7 it follows that the Weil polyhedron is always a domain of holomorphy. This result could also be obtained from other considerations.

3. Conservation of properties of the kernel function for its continuation. The present subsection is devoted to the implications which make it possible to continue the kernel function of a domain. The question of when such a continuation is indeed possible will be considered in the next subsection.

THEOREM 5.8 (Sommer-Mehring [1], (I)). *Let D and G be domains over the space C^n of the variables $z_1, \cdots, z_n$, while the domain D is single-sheeted and bounded and the domain G has no interior branch point and $D < G$. Then if the kernel function $K_D(z, \bar{z})$ can be continued to the domain G as a real analytic function, it follows that:*

1) *Every function $f \in L^2(D)$ can be holomorphically continued to the domain G.*

2) *If $\{\phi_i(z),\ i = 1, 2, \cdots\}$ is a closed orthonormal system of functions in the domain D, then the equation* (1.37)

$$K_D(z, \bar{\zeta}) = \sum_{i=1}^{\infty} \varphi_i(z)\, \overline{\varphi_i(\zeta)}$$

holds in the whole domain $G_z \times G_{\bar{\zeta}}$; for every function $f \in L^2(D)$ the expansion (1.42)

$$f(z) = \sum_{i=1}^{\infty} a_i \varphi_i(z)$$

remains valid in the whole domain G.

3) *The function $K_D(z, \bar{\zeta})$ is holomorphic with respect to the variables z and $\bar{\zeta}$ in the domain $G_z \times G_{\bar{\zeta}}$.*

4) *The integral representation*

$$f(z) = \int_{D_\zeta} K_D(z, \bar{\zeta})\, f(\zeta)\, dv_\zeta \tag{1.72}$$

(Theorem 4.7) is valid at all points $z \in G$.

5) *At all points* $z \in G$ *the equation* (1.49)

$$K_D(z, \bar{z}) = \max |f(z)|^2$$

holds for functions $f(z) \in L^2(D)$ *satisfying the condition* $\|f\|_D \leq 1$.

PROOF. We will carry out the proof for the case $n = 2$.

1) We are given that the kernel function $K_D(z, \bar{z})$ can be continued to any point $z^* \in G$ as a real analytic function, i.e., it can be represented in the neighborhood of that point as the series

$$K_D(z, z) = \sum_{k,l,m,p=0}^{\infty} K^*_{kl\bar{m}\bar{p}} (z_1 - z_1^*)^k (z_2 - z_2^*)^l (\bar{z}_1 - \bar{z}_1^*)^m (\bar{z}_2 - \bar{z}_2^*)^p, \tag{1.73}$$

where

$$K^*_{kl\bar{m}\bar{p}} = \frac{1}{k!\,l!\,m!\,p!} \left.\frac{\partial^{k+l+m+p} K_D(z, \bar{z})}{\partial z_1^k \, \partial z_2^l \, \partial \bar{z}_1^m \, \partial \bar{z}_2^p}\right|_{z=z^*} = \frac{1}{k!\,l!\,m!\,p!} \left.\frac{\partial^{k+l+m+p} K_D(z, \bar{\zeta})}{\partial z_1^k \, \partial z_2^l \, \partial \bar{\zeta}_1^m \, \partial \bar{\zeta}_2^p}\right|_{(z,\bar{\zeta})=(z^*,\bar{z}^*)} \tag{1.74}$$

Here ζ_1 and ζ_2 are new complex variables. Replacing $\bar{z}_1$ and $\bar{z}_2$ by $\bar{\zeta}_1$ and $\bar{\zeta}_2$ in the expression (1.42), we obtain a power series in the variables z_1, z_2, $\bar{\zeta}_1$, $\bar{\zeta}_2$ which converges absolutely and uniformly in the closed polycylinder $\overline{\mathfrak{S}^*_z \times \mathfrak{S}^*_{\bar{\zeta}}}$. Here the bicylinder $\mathfrak{S}^*_z = \{|z_j - z_j^*| < R^*, \ j = 1, 2\}$ and R^* is a certain positive number. In the neighborhood of a point $(z^*, \bar{z}^*)$ of the "diagonal" section of the domain $G_z \times G_{\bar{\zeta}}$ by the planes $z_1 = \zeta_1$ and $z_2 = \zeta_2$ this series defines a holomorphic function of the variables z_1, z_2, $\bar{\zeta}_1$, $\bar{\zeta}_2$, which coincides there with the kernel function $K_D(z, \bar{\zeta})$.

We consider a path which lies in the domain G and connects a point $z^{(0)} \in D$ with a point $z^* \in G$; let L be the image of this path on the diagonal section of the domain $G_z \times G_{\bar{\zeta}}$. Let us cover the path L by a finite set of polycylinders $\mathfrak{S}^{(\nu)}_z \times \mathfrak{S}^{(\nu)}_{\bar{\zeta}}$, $\nu = 0, 1, \cdots, r$, each having its center at the point $(z^{(\nu)}, z^{(\nu)}) \in L$. Here the bicylinder $\mathfrak{S}^{(\nu)}_z = \{|z_j - z_j^{(\nu)}| < R_j^{(\nu)}, \ j = 1, 2\}$ and the R_j are certain positive numbers, $\mathfrak{S}^{(r)}_z = \mathfrak{S}^*_z$, and the point $z^{(\nu)} \in \mathfrak{S}^{(\nu-1)}_z$ for $\nu = 1, \cdots, r$. The numbers $R^{(\nu)}$ are chosen in such a way that every series of the form (1.73) representing the kernel function $K_D(z, \bar{\zeta})$ in the neighborhood of the point $z^{(\nu)}$ converges in the closed polycylinder $\overline{\mathfrak{S}^{(\nu)}_z \times \mathfrak{S}^{(\nu)}_{\bar{\zeta}}}$.

By the definition of the kernel function itself $K_D(z, \bar{\zeta})$ can be represented in some neighborhood U_0 of the point $(z^{(0)}, \bar{z}^{(0)})$ by the absolutely and uniformly convergent series

$$K_D(z, \bar{\zeta}) = \sum_{i=1}^{\infty} \varphi_i(z) \overline{\varphi_i(\zeta)}, \tag{1.37}$$

where $\{\phi_i(z),\ i = 1, 2, \cdots\}$ is some closed orthonormal system of functions in the domain D. From equations (1.37) and (1.74) it follows that at the point $(z^{(0)}, \bar{z}^{(0)})$ we have

$$K^{(0)}_{kl\bar{m}\bar{p}} = \sum_{i=1}^{\infty} \varphi^{(0)}_{i,\,kl} \overline{\varphi^{(0)}_{i,\,\bar{m}\bar{p}}},$$

where

$$\varphi^{(0)}_{i,\,kl} = \frac{1}{k!\,l!} \left.\frac{\partial^{k+l}\varphi_i(z)}{\partial z_1^k \, \partial z_2^l}\right|_{z=z^{(0)}}.$$

From the convergence of the series (1.73) with center at the point $z^{(0)}$ it follows that the series

$$\sum_{k,l,m,p=0} \left| K^{(0)}_{kl\bar{m}\bar{p}} \right| R^{k+l+m+p} = \sum_{s=0}^{\infty} \Bigl(\sum_{k+l+m+p=s} \left| K^{(0)}_{kl\bar{m}\bar{p}} \right| \Bigr) R^s$$

converges for $0 < R < R_0$ and therefore that there exists a number $A > 0$ such that for all s

$$\sum_{k+l+m+p=s} \left| K^{(0)}_{kl\bar{m}\bar{p}} \right| < \frac{A}{R^s}.$$

For $k = m,\ l = p$ we obtain

$$K^{(0)}_{kl\bar{k}\bar{l}} = \sum_{i=1}^{\infty} \left| \varphi^{(0)}_{i,\,kl} \right|^2,$$

$$\sum_{k+l=s} \left| K^{0}_{kl\bar{k}\bar{l}} \right| = \sum_{k+l=s} K^{0}_{kl\bar{k}\bar{l}} = \sum_{k+l=s} \sum_{i=1}^{\infty} \left| \varphi^{(0)}_{i,\,kl} \right|^2$$

$$= \sum_{i=1}^{\infty} \sum_{k+l=s} \left| \varphi^{(0)}_{i,\,kl} \right|^2 < \sum_{k+l+m+p=2s} \left| K^{0}_{kl\bar{m}\bar{p}} \right| < \frac{A}{R^{2s}}.$$

Hence it is seen that

$$\sum_{k+l=s} \left| \varphi^{(0)}_{i,\,kl} \right|^2 < \frac{A}{R^{2s}} \tag{1.75}$$

for each of $s = 1, 2, 3, \cdots$. By making use of the Bunjakovskiĭ-Schwarz inequality we further obtain

$$\sum_{k+l=s} \left| \varphi^{(0)}_{i,\,kl} \right| < \sqrt{s+1}\, \frac{\sqrt{A}}{R^s}.$$

In the bicylinder $\mathfrak{S}_z^{(0)}$ we consider the series

$$\varphi_i(z) = \sum_{k,l=0}^{\infty} \varphi_{i,kl}^{(0)} \left(z_1 - z_1^{(0)}\right)^k \left(z_2 - z_2^{(0)}\right)^l. \tag{1.76}$$

For $|z_1 - z_1^{(0)}| < \theta R$, $|z_2 - z_2^{(0)}| < \theta R$, $0 < \theta < 1$ we have

$$\sum_{k,l=0}^{\infty} \left| \varphi_{i,kl}^{(0)} \left(z_1 - z_1^{(0)}\right)^k \left(z_2 - z_2^{(0)}\right)^l \right|$$
$$< \sum_{s=0}^{\infty} \left(\sum_{k+l=s} |\varphi_{i,kl}^{(0)}| \theta^s R^s \right) < \sum_{s=0}^{\infty} \sqrt{s+1}\, \sqrt{A}\, \theta^s.$$

Hence it follows that the series (1.76) converges absolutely and uniformly in the bicylinder $\mathfrak{S}_z^{(0)}$ and that the function $\phi_i(z)$ is holomorphic in this bicylinder.

Repeating this argument successively for the bicylinders $\mathfrak{S}_1, \cdots, \mathfrak{S}_r = \mathfrak{S}^*$, we arrive at the conclusion that the functions $\phi_i(z)$ are holomorphic at an arbitrary point $z^* \in G$, i.e., they are holomorphic in the domain G.

It is well known (see Remark 3 to Theorem 4.5) that any function $f \in L^2(D)$, after proper normalization, may be included in a closed orthonormal system as the function $\phi_1(z)$. From this our first assertion follows.

2) We now show that the series $\Sigma_{i=1}^{\infty} |\phi_i(z)|^2$ converges uniformly in the bicylinder $\mathfrak{S}_z^{(0)}$. Indeed, by using the Bunjakovskiĭ-Schwarz inequality we find that for $|z_j - z_j^{(0)}| < \theta R$, $j = 1, 2$,

$$|\varphi_i(z)| = \left| \sum_{s=0}^{\infty} \sum_{k+l=s} \varphi_{i,kl}^{(0)} \left(z_1 - z_1^{(0)}\right)^k \left(z_2 - z_2^{(0)}\right)^l \right|$$
$$\leqslant \sum_{s=0}^{\infty} \sum_{k+l=s} |\varphi_{i,kl}^{(0)}| R^s \theta^s \leqslant \sum_{s=0}^{\infty} \sqrt{(s+1) \sum_{k+l=s} |\varphi_{i,kl}^{(0)}|^2}\, R^s \theta^s.$$

From the inequality (1.75) it follows that

$$\sum_{k+l=s} |\varphi_{i,kl}^{(0)}|^2 = \frac{A_i}{R^{2s}},$$

where $A_i \geq 0$ and $\Sigma_{i=1}^{\infty} A_i < A$. Therefore we have

$$|\varphi_i(z)| \leqslant \sum_{s=0}^{\infty} \sqrt{s+1}\, \sqrt{A_i}\, \theta^s < \sqrt{A_i}\, \frac{1}{(1-\theta)^2}$$

or

$$|\varphi_i(z)|^2 < A_i \frac{1}{(1-\theta)^4}.$$

Hence, since $\Sigma_{i=1}^{\infty} A_i < A$, it follows that the series $\Sigma_{i=1}^{\infty} |\phi_i(z)|^2$ converges absolutely and uniformly in the bicylinder $\mathfrak{S}_z^{(0)}$ and that the series

$$\sum_{i=1}^{\infty} \phi_i(z)\overline{\phi_i(\zeta)}$$

behaves in a similar fashion in the polycylinder $\mathfrak{S}_z^{(0)} \times \mathfrak{S}_{\overline{\zeta}}^{(0)}$.

Repeating this argument successively for the polycylinders $\mathfrak{S}_z^{(\nu)} \times \mathfrak{S}_{\overline{\zeta}}^{\nu}$, $\nu = 1, \cdots, r$, we establish that the series $\Sigma_{i=1}^{\infty} \phi_i(z)\overline{\phi_i(\zeta)}$ converges absolutely and uniformly in the domain $G_z \times G_{\overline{\zeta}}$ and that it defines a continuation of the function $K_D(z, \overline{\zeta})$ in that domain. The second part of our assertion follows from the uniform convergence of the series $\Sigma_{i=1}^{\infty} |\phi_i(z)|^2$ in the domain G and the convergence of the series $\Sigma_{i=1}^{\infty} |a_i|^2$.

3) The third assertion of the theorem follows immediately from the first two.

4) Let a point $z \in G$. Then the series $\Sigma_{i=1}^{\infty} |\phi_i(z)|^2$ converges, and therefore the function $\overline{K_D(z, \overline{\zeta})} \in L^2(D_\zeta)$ and the integral $\int_{D_\zeta} K_D(z, \overline{\zeta}) f(\zeta) dv_\zeta$ exists for any function $f \in L^2(D)$. In this connection the equality

$$\int_{D_\zeta} K_D(z, \overline{\zeta}) f(\zeta)\, dv_\zeta$$

$$= \sum_{i=1}^{n} \varphi_i(z) \int_{D_\zeta} \overline{\varphi_i(\zeta)} f(\zeta)\, dv_\zeta = \sum_{i=1}^{\infty} a_i \varphi_i(z) = f(z)$$

holds. Thus our assertion is proved.

5) We have already shown above that $\overline{K_D(z, \overline{\zeta})} \in L^2(D_\zeta)$; it is easy to see that $\|K_D(z, \overline{\zeta})\|_{D_\zeta}^2 = K_D(z, \overline{z})$ (here we always have $z \in G$). From this and the integral representation (1.72) we find, by the use of the Bunjakovskiĭ-Schwarz inequality, that

$$|f(z)|^2 \leqslant K_D(z, \overline{z}) \|f\|_D^2.$$

Therefore for every function $f \in L^2(D)$ satisfying the condition $\|f\|_D \leq 1$ at all points $z \in G$ the inequality

$$K_D(z, \overline{z}) \geqslant |f(z)|^2$$

holds.

This inequality becomes an equality at each fixed point $z \in G$ for the function

$g(\zeta) = \overline{K_D(z, \overline{\zeta})}\,(K_D(z, \overline{z}))^{-1/2}$. From this our assertion follows.

SUPPLEMENT (TO THEOREM 5.8) 1. *Let D_1, D_2 and G be domains over the space C^n, while the domains D_1 and D_2 are single-sheeted and bounded, the domain G has no interior branch point and $D_1 \subset D_2 \subset G$. In this case, if the kernel functions $K_{D_1}(z, \overline{z})$ and $K_{D_2}(z, \overline{z})$ are continuable to the domain G as real analytic functions, then at points $z \in G$ the inequality* (1.64)

$$K_{D_1}(z, \overline{z}) \geq K_{D_2}(z, \overline{z})$$

is valid.

The proof may be reduced to the application of the last assertion of Theorem 5.8.

• SUPPLEMENT (TO THEOREM 5.8) 2. *In the case that the assumptions of Theorem 5.8 are satisfied, the form $\Sigma_{k,l=1}^n T_{k\overline{l}}(z, \overline{z})dz_k\overline{dz}_l$ defining the Bergman metric in the domain D is continuable to the domain G: its coefficients remain real analytic functions and the form itself remains hermitian and positive definite. In the domain G it defines a Riemann metric, invariant under biholomorphic mappings of that domain.*

PROOF. In forming a closed orthonormal system $\{\phi_\nu(z),\ \nu = 1, 2, \cdots\}$ we take $\phi_1(z) = K_D(z, \overline{z}^{(0)}) \cdot (K_D(z^{(0)}, \overline{z}^{(0)}))^{-1/2}$, where $z^{(0)} \in G$. Then $K_D(z^{(0)}, \overline{z}^{(0)}) = |\phi_1(z^{(0)})|^2$ and accordingly $\phi_\nu(z^{(0)}) = 0$ for $\nu = 2, 3, \cdots$.

It is easy to see that for our choice of the function $\phi_1(z)$ the product $(f, \phi_1)_D$ for functions $f \in L^2(D)$ is equal to zero if and only if $f(z^{(0)}) = 0$. This condition is necessary, since the function $f(z)$ orthogonal to the function $\phi_1(z)$ may, after proper normalization, be taken as the function $\phi_2(z)$. Sufficiency of the condition is proved by contradiction: suppose that $f(z^{(0)}) = 0$, while $(f, \phi_1)_D = a \neq 0$. Then $(f - a\phi_1, \phi_1) = 0$ and, accordingly, $f(z^{(0)}) - a\phi_1(z^{(0)}) = 0$. Hence, as $\phi_1(z^{(0)}) \neq 0$, we conclude that $a = 0$.

From formula (1.70) and by using the second assertion of Theorem 5.8 we find that

$$ds^2\big|_{z=z^{(0)}} = \frac{1}{K_D(z^{(0)}, \overline{z}^{(0)})} \sum_{i=2}^{\infty} |d\varphi_i|^2\big|_{z=z^{(0)}}. \tag{1.77}$$

From this our assertion follows; it only remains for us to establish that the form (1.77) does not become zero for $\Sigma_{k=1}^n |dz_k| \neq 0$. This is evident if we take $\phi_2(z) = a_1(z_1 - z_1^{(0)})$, where $a \neq 0$.

REMARK. The theorem we have just proved implies that if the domain of

holomorphy D is the holomorphy hull of a certain subdomain $D_0 \neq D$, then, in this domain, integral representations of the form (1.72) with kernels $K_{D_0}(z, \bar{z})$ and $K_D(z, \bar{z})$ are possible. The former is useful for functions of a wider class than the latter, since we know from the first assertion of Theorem 5.8 that $L^2(D_0) \supset L^2(D)$.

4. Continuation of the kernel function of a domain. Exterior holomorphy hull of a domain.

DEFINITION (hull of a domain relative to its kernel function). By the hull of a single-sheeted domain D relative to its kernel function over the space C^n we mean the largest domain over the space C^n, containing the domain D, to which the kernel function $K_D(z, \bar{z})$ is continuable as a real analytic function. This hull is denoted by the symbol $\mathbf{K}(D)$.

In view of the corollary of Theorem 4.1, the kernel function $K_D(z, \bar{\zeta})$ of a bounded domain D is holomorphic as a function of the variables z and $\bar{\zeta}$ in the domain $D_z \times D_{\bar{\zeta}}$. Hence it follows that it is holomorphic in the domain $H(D_z \times D_{\bar{\zeta}}) = H(D) \times H(D_{\bar{\zeta}})$ (the last equality is due to Theorem 3.11, (I)), and that the kernel function $K_D(z, \bar{z})$ is continuable to the domain $H(D)$ as a real analytic function. Thus we arrive at the conclusion that

$$H(D) \subset \mathbf{K}(D). \tag{1.78}$$

The further investigation of the domain $\mathbf{K}(D)$ requires consideration of the holomorphy hull of the domain D.

DEFINITION. The domain defined as the intersection of all domains of holomorphy, each containing strictly inside itself a domain D over the space P^n (the intersection is taken relative to the domain D; see §8.5, Chapter II, (I)), is called the *exterior holomorphy hull* of the domain D and is denoted by the symbol $N(D)$. If $N(D) \neq H(D)$, the domain $N(D)$ is called the *complementary holomorphy hull* of the domain D.

The following examples assure the existence of domains belonging to the complementary holomorphy hull.

EXAMPLE 1. Consider the bicircular domain $D = \{|w| < 1,\ |z| < |w|\} \subset C^2_{w,z}$. We observe first of all that $H(D) = D$. This follows, in view of Theorem 13.6, (I), from the fact that for every boundary point $P_0(w_0, z_0)$ of the domain D one can find a function $f(w, z)$ holomorphic in the domain D and singular at that point. We set

$$f_{P_0}(w, z) = \begin{cases} (w\bar{w}_0 - z\bar{z}_0)^{-1} & \text{for} \quad |w_0| = |z_0|, \\ (w - w_0)^{-1} & \text{for} \quad |w_0| = 1. \end{cases}$$

Evidently the function $f_{P_0}(w, z)$ possesses the required properties and therefore indeed $H(D) = D$.

Let D^* be an arbitrary domain of holomorphy containing the domain D strictly inside itself. Then the point $(0, 0) \in D^*$ and consequently there exists a number $R > 0$ such that $\mathfrak{S}_R = \{|w| \leq R, \ |z| \leq R\} \subset D^*$. Thus we have $D \cup \mathfrak{S}_R \Subset D^*$. Every function holomorphic in the domain $D \cup \mathfrak{S}_R$ will be automatically holomorphic in the smallest of those complete bicircular domains centered at the origin of the coordinates which contain the domain $D \cup \mathfrak{S}_R$. This smallest complete domain is evidently the bicylinder $\mathfrak{S} = \{|w| < 1, \ |z| < 1\}$.

Thus $D^* \subset \mathfrak{S}$. On the other hand, for any point $P \notin \mathfrak{S}$ one can find a domain D^* not containing that point (for example, the bicylinder $\{|w| < 1 + 1/m, \ |z| < 1 + 1/m\}$, where m is a sufficiently large number). Therefore the bicylinder $\mathfrak{S} = N(D)$ is the complementary holomorphy hull of the domain D.

EXAMPLE 2. Consider the deleted disk $D = \{0 < |z| < 1\}$. In this case $H(D) = D$, since every domain on the plane C^1 coincides with its holomorphy hull. On the other hand, in constructing the exterior holomorphy hull, we make up the intersection of domains of holomorphy each containing the domain D strictly inside itself. For this reason, in the case in question, the exterior holomorphy hull is the complete disk $\{|z| < 1\}$; hence it is shown to be the complementary holomorphy hull of the domain D.

Behnke and Stein [1] established a series of sufficient conditions for the presence of the complementary holomorphy hull of a domain.

Almost all properties of the holomorphy hulls (see §13, Chapter II, (I)) are transferable to the exterior holomorphy hulls. However, in order to derive the properties of the hull $N(D)$, we have to consider functions holomorphic in the closed domain $\bar{D}$ (not in the domain D, as in the case of the hull $H(D)$). In particular the following proposition (used below) is valid:

Let H be a domain of holomorphy containing the domain D strictly inside itself, and let r be the minimal boundary distance of the domain D in the domain H. Then the minimal boundary distance of the hull $N(D)$ in the domain H is also equal to r.

The proof of this proposition is analogous to that of Theorem 13.4, (I).

We can now formulate a fundamental result of the theory under discussion:

THEOREM 5.9. *If D is a bounded single-sheeted domain over the space C^n, then*

$$H(D) \subset \mathbf{K}(D) \subset N(D).$$

PROOF. The inclusion relation $H(D) \subset \mathbf{K}(D)$ has already been established and so we need only show that $\mathbf{K}(D) \subset N(D)$. Suppose that, contrary to our assertion, there exists a point $z^{(0)} \in \partial[N(D)] \cap \mathbf{K}(D)$. Then in some neighborhood U of this point: 1) the function $K_D(z, \bar{z})$ is bounded; 2) there are points not belonging to the domain $N(D)$. Consequently there exists a domain of holomorphy $H \Supset D$ whose boundary intersects the neighborhood U. Let $2r$ be the minimal boundary distance of the domain D in the domain H. Then, as we have stated above, the minimal boundary distance of the domain $N(D)$ in the domain H is also equal to $2r$.

We consider the open set consisting of points of the domain H with boundary distance greater than r. Let H_r be the connected component of this set which contains the domain $N(D)$. By virtue of the holomorphic convexity of the domain H, for every point $z' \in \partial H_r$ one can find a function $f(z)$ holomorphic in this domain such that

$$|f(z')| > 1, \tag{1.79}$$

and at the same time

$$\sup |f(N(D))| < 1.$$

From continuity considerations it further follows that the inequality (1.79) still holds in some neighborhood $S' \subset \partial H_r$ of that point z'. The boundary ∂H_r is a closed set and therefore, by Borel's lemma, it can be covered by a finite set of similar neighborhoods $S_j \subset \partial H_r$ of the points $z_j \in \partial H_r$ $(j = 1, \cdots, N)$. Let $f_j(z)$ be corresponding functions holomorphic in the domain H and satisfying the conditions

$$|f_j(z)|_{z \in S_j} > 1, \quad \sup |f_j(N(D))| < 1.$$

Now we consider the set of points of the domain H for which

$$|f_j(z)| < 1, \quad j = 1, \cdots, N.$$

Among connected components of this set, there is one that contains the domain $N(D)$ in its interior. We shall denote it by Δ. Evidently 1) $N(D) < \Delta \ll H$.

2) The boundary $\partial\Delta$ consists of the hypersurfaces $|f_j(z)| = 1$, $j = 1, \cdots, N$.[1] The boundary $\partial\Delta$ intersects the neighborhood U.

Let a point $z^* \in (U \cap \partial\Delta)$. In view of the first supplement to Theorem 5.8 at points $z \in (U \cap \Delta)$, we have

$$K_D(z, \bar{z}) \geqslant K_\Delta(z, \bar{z}). \tag{1.80}$$

By the remark to Theorem 5.7 the function $K_\Delta(z, \bar{z})$ must increase indefinitely on the approach of the point z to the point $z^* \in \partial\Delta$, which, because of the inequality (1.80), is inconsistent with the boundedness of the function $K_D(z, \bar{z})$ in the neighborhood U of the point $z^{(0)}$.

This contradiction forces us to reject our supposition. The theorem is proved.

COROLLARY. *If a bounded single-sheeted domain D over the space C^n has no complementary holomorphy hull, i.e., if $N(D) = H(D)$, then* $\mathbf{K}(D) = H(D)$.

In this case the kernel function $K_D(z, \bar{z})$ *turns out to be unbounded in any neighborhood of each point of the boundary* $\partial H(D)$. One can prove (see Bremermann [2], (I)) under some auxiliary conditions, that it becomes indefinitely large on the approach of the point $z \in H(D)$ to the boundary $\partial H(D)$ of the hull $H(D)$.

EXAMPLE. We again consider the deleted disk $D = \{0 < |z| < 1\}$. We have seen above that in this case $H(D) = D$ and $N(D) = \{|z| < 1\}$. Every function $f(z) \in L^2(D)$ may be represented in the domain D by the Laurent series

$$f(z) = \sum_{n=-\infty}^{n=\infty} a_n z^n.$$

Therefore for any integer ν

$$\infty > \|f\|_D^2 > \int_{\theta < |z| < 1-\theta} |f(z)|^2 dv$$
$$= \sum_{-\infty}^{\infty} |a_n|^2 \int_{\theta < |z| < 1-\theta} |z|^{2n} dv \geqslant |a_\nu|^2 \int_{\theta < |z| < 1-\theta} |z|^{2\nu} dv.$$

Hence, so long as $\|f\|_D^2$ is finite, it follows that

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[1] To obtain the polyhedron Δ we first replace each part S_j of the boundary ∂H_r by the hypersurface $\{|f_j(z)| = 1\}$. Then we consider the domain which is bounded by these hypersurfaces and encloses the domain $N(D)$.

$$\lim_{\theta\to 0}\left(|a_\nu|^2 \int\limits_{\theta<|z|<1-\theta} |z|^{2\nu}\,dv\right) < \infty,$$

and, accordingly, $a_\nu = 0$ for $\nu < 0$. Thus if $f \in L^2(D)$, then $f \in L^2(N(D))$. Moreover it is obvious that there is no difference between orthogonality of two functions holomorphic in the domain $N(D)$ with respect to the domains D and $N(D)$. By what has been said, the closed orthonormal systems of functions for the domains D and $N(D)$ coincide with each other and so we have

$$K_D(z, \bar{z}) = K_{N(D)}(z, \bar{z}) = \frac{1}{\pi(1 - z\bar{z})^2}.$$

Hence it follows that the kernel function $K_D(z, \bar{z})$ remains a real analytic function at the boundary point $z = 0$ of the domain of holomorphy D.

5. Bergman's results concerning the behavior of the kernel function on the boundary of a domain. Already in his first work dealing with the kernel function, S. Bergman investigated the behavior on the boundary of a domain D of the kernel function $K_D(z, \bar{z})$ and its related quantities. He showed that $\lim K_D(z, \bar{z})$, for the approach of the point $z \in D$ to the boundary point $\zeta \in \partial D$, depends on the structure of the boundary ∂D of the domain D and on the way in which the point z approaches the point ζ. By comparing Theorems 5.5–5.7 and 5.9 with Bergman's theorems it is easily seen that the latter are more precise, though they are concerned with a more restricted class of domains. We mention here (without proof) one of his theorems.

THEOREM 5.10 (Bergman [5]). *Let D be a bounded domain of the space C^2, while*:

1) *its boundary ∂D is a hypersurface of the class $\mathcal{C}^2$;*

2) *at a point $\zeta \in \partial D$ the hypersurface ∂D is convex in the sense of Levi and one of the analytic surfaces passing through that point lies entirely outside the domain D;*

3) *a point z approaches the point ζ, remaining in the interior of the cone formed by rays which emanate from the point ζ and make an angle not larger than α with the outer normal N to the hypersurface ∂D at the point ζ. Here α is a number satisfying the inequality $0 < \alpha < \pi/2$.*

Then

$$\lim_{z\to\zeta} n^3 K_D(z, \bar{z}) = A.$$

Here n is the projection of the segment $z\zeta$ onto the normal N and A is some precisely determined constant.

If the hypersurface ∂D is defined in the neighborhood of the point ζ by the equation $\Phi = 0$, while the domain D belongs to this hypersurface on the side $\Phi < 0$ and ζ is its ordinary point, then in the case considered in Theorem 5.10, Levi's determinant $L(\Phi)|_{\zeta} > 0$.

Different results are obtained if at such a point ζ Levi's determinant is less than or equal to zero. In the first of these cases the kernel function $K_D(z, \bar{z})$ remains bounded for $z \longrightarrow \zeta$. In the second case it increases indefinitely for $z \longrightarrow \zeta$, but the order of growth to infinity is not equal to three ($\lim n^2 K_D(z, \bar{z})$ turns out to be finite). However, here it is necessary to consider some method that is different from the above one of letting the point z approach the point ζ.

Analogous limiting relations hold for various quantities related to the Bergman metric (see §25.4, Chapter V).

§6. SEQUENCES OF DOMAINS. PROBLEM OF CONVERGENCE OF HOLOMORPHY HULLS

1. **Sequence of domains.** We consider a sequence of domains D_ν, $\nu = 1, 2, \cdots$ over the space P^n, each intersecting with some domain D_0. Let B_μ be an intersection of some infinite subset of these domains relative to the domain D_0. Then a domain D is the *kernel of the sequence of domains* $\{D_\nu\}$ relative to the domain D_0 when it contains inside itself all domains B_μ, and is contained inside any other domain having this property. If every subsequence of the sequence $\{D_\nu\}$ also has the domain D as its kernel, then that domain D is said to be the *limit of the sequence of domains* $\{D_\nu\}$. In this case we write: $\lim_{\nu\to\infty} D_\nu = D$. Evidently if the intersection of a set of domains $\{D_\nu\}$ defines a certain domain, then the set of these domains may have certain kernels and certain limit domains.

Our definition of single-sheeted domains signifies that the domain D is the limit of the sequence of domains $\{D_\nu\}$ if and only if each point $z \in D$ with its neighborhood $U_z \subset D$ lies inside the intersection of almost all domains D_ν (for $\nu > \nu_z$) or, alternatively, every point $z \notin D$ lies inside the intersection of only a finite set of domains D_ν.

If $\lim_{\nu\to\infty} D_\nu = D$ one says that the sequence of domains $\{D_\nu\}$ *approximates the domain* D. If $\lim_{\nu\to\infty} D_\nu = D$, and if always $D_\nu > D_{\nu+1}$, then one says that the sequence $\{D_\nu\}$ approximates the domain D "from the outside"; if always

$D_\nu < D_{\nu+1}$, one speaks, instead, of approximation "from the inside".

An important role is also played by a *principal sequence of domains*. This name is given to a sequence of domains $\{D_\nu\}$ for which always $D_\nu \gg D_{\nu+1}$ or $D_\nu \ll D_{\nu+1}$.

All of these concepts are applicable not only to domains but to arbitrary open sets. Later we shall use this fact.

We consider a sequence of domains $\{D_\nu\}$ which approximates a domain D and a sequence of holomorphy hulls $\{H(D_\nu)\}$. The question arises: when does $\lim_{\nu\to\infty} H(D_\nu) = H(\lim_{\nu\to\infty} D_\nu) = H(D)$? This is the problem of the convergence of holomorphy hulls. Evidently it can be stated as the question of the continuity conditions for the function $H(D)$ which defines the hull with respect to the domain D.

Let $\{D_\nu\}$ be a principal sequence of domains which approximates the domain D from the outside. Then $\lim_{\nu\to\infty} H(D_\nu)$ exists and coincides with the intersection of the domains $H(D_\nu)$ (since we have always $H(D_{\nu+1}) < H(D_\nu)$) and thus it is a domain of holomorphy. This limit domain does not depend on the choice of the sequence $\{D_\nu\}$. Indeed, if $\{D_\mu^*\}$ is another sequence of domains which approximates the domain D from the outside, then for every number μ one can find a number ν such that $D_\mu^* \ll D_\nu$, and vice versa. Hence it is easily concluded that in the case in question $\lim_{\nu\to\infty} H(D_\nu) = N(D)$, where $N(D)$ is the complementary holomorphy hull of the domain D, since one can always specially construct a principal sequence of domains $\{D_\nu^0\}$ such that

$$\lim_{\nu\to\infty} D_\nu^0 = D, \quad \lim_{\nu\to\infty} H(D_\nu^0) = N(D).$$

Thus in the case under consideration $\lim_{\nu\to\infty} H(D_\nu) = H(D)$ only if $N(D) = H(D)$, i.e., only if the domain D has no complementary holomorphy hull.

2. Sequences of domains of holomorphy. Now we are going to solve the problem of the convergence of domains of holomorphy for single-sheeted domains. In this case the following theorem plays an important role:

THEOREM 6.1. [1] *If a sequence of domains of holomorphy $\{D_\nu\}$ converges from the inside to a bounded domain $D \subset C^n$, then the latter is also a domain of holomorphy.*

This theorem comes from another weaker theorem which is formulated as follows:

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[1] See Behnke-Stein [1], where this theorem is proved for domains of more general type.

THEOREM 6.2. *If a principal sequence of domains of holomorphy* $\{D_\nu\}$ *converges from the inside to a bounded domain* $D \subset C^n$, *then the latter is also a domain of holomorphy.*

We will divide the proof of this theorem into several lemmas. The proof is essentially based on the results of K. Oka mentioned in §2 of the present chapter.

LEMMA 1. *If a bounded domain* $D \subset C^n$ *is approximated from the inside by a principal sequence of domains* D_ν $(\nu = 1, 2, \cdots)$ *which are holomorphically convex relative to the domain* D, *then this last domain is a domain of holomorphy.*

PROOF. Suppose that our assertion is not true. Then there exist 1) a domain $\mathfrak{D} \Subset D$ with the minimal boundary distance [1] in the domain D, equal to $\rho = 2r_0$, and 2) a sequence of points $M_k \in D$, where $k = 1, 2, \cdots$, $\lim_{k\to\infty} M_k = M_0$, the point M_0 being in ∂D, such that for every function $f(z) \in \mathfrak{H}_D$ and arbitrary k we have

$$\sup|f(\mathfrak{D})| \geqslant |f(M_k)|.$$

Consider a domain $D_0 = \mathfrak{D}^{(r_0)} \Subset D$. Then, in view of Theorem 11.1, (I), for every function $f(z) \in \mathfrak{H}_D$ and arbitrary k we have

$$\sup|f(D_0)| \geqslant \sup|f(S(M_k, r_0))|$$

and therefore

$$\sup|f(D_0)| \geqslant \sup|f(S(M_0, r_0))|. \tag{1.81}$$

On the other hand, so long as $\lim_{\nu\to\infty} D_\nu = D$, there exists a number ν_0 such that $D_0 \Subset D_\nu$ for $\nu > \nu_0$. The domains D_ν are convex relative to the family of functions $\mathfrak{H}_D$. Therefore for $\nu > \nu_0$ and a point $z \in D_\nu \setminus \hat{D}_{0,\nu}$, where $\hat{D}_{0,\nu}$ is $\mathfrak{H}_D$-convex hull of the domain D_0 in the domain D_ν, we have

$$\sup|f(D_0)| < |f(z)|. \tag{1.82}$$

Since $\lim_{\nu\to\infty} D_\nu = D$, the intersection $\partial D_\nu \cap S(M_0, r_0) \neq \emptyset$ for sufficiently large values of ν and the point $z \in D_\nu \setminus \hat{D}_{0,\nu}$ must be in the polycylinder $S(M_0, r_0)$. This is impossible, because the inequalities (1.81) and (1.82) are incompatible. Thus we must reject our supposition. The lemma is proved.

LEMMA 2. *If a bounded domain* $D \subset C^n$ *is approximated from the inside by a principal sequence of domains* D_ν $(\nu = 1, 2, \cdots)$, *while each domain* D_ν *is*

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[1] Here we understand the distance in the sense stated in §11, Chapter II, (I).

holomorphically convex relative to the domain $D_{\nu+1}$, *then* D *is a domain of holomorphy.*

PROOF. Let the domain $\mathfrak{D}_\nu \Subset D_\nu$. Since the domain D_ν is holomorphically convex relative to the domain $D_{\nu+1}$, to each point $M \in D_\nu \setminus \hat{\mathfrak{D}}_{\nu,\nu+1}$, where $\hat{\mathfrak{D}}_{\nu,\nu+1}$ is the $\mathfrak{H}_{D_{\nu+1}}$-convex hull of the domain $\mathfrak{D}_\nu$, there corresponds a function $f_M(z) \in \mathfrak{H}_{D_{\nu+1}}$ for which

$$|f_M(M)| > \sup |f_M(\mathfrak{D}_\nu)|.$$

We take a number $\delta > 0$ and a sequence of numbers $\epsilon_\rho > 0$ $(\rho = 1, 2, \cdots)$ such that

$$|f_M(M)| > \sup |f_M(\mathfrak{D}_\nu)| + \delta, \tag{1.83}$$

$\Sigma_{\rho=1}^\infty \epsilon_\rho < \delta/2$. By the hypothesis of the lemma the function $f_M(z)$ can be uniformly approximated in the domain $D_{\nu+1}$ by functions holomorphic in the domain $D_{\nu+2}$. Suppose that a function $f_{M,1}(z) \in \mathfrak{H}_{D_{\nu+2}}$, and for $z \in D_\nu$

$$|f_{M,1}(z) - f_M(z)| < \varepsilon_1.$$

Further we choose a function $f_{M,2}(z) \in \mathfrak{H}_{D_{\nu+3}}$, such that for $z \in D_{\nu+1}$

$$|f_{M,2}(z) - f_{M,1}(z)| < \varepsilon_2.$$

Reasoning in this way we can define a sequence of functions $f_{M,\rho}(z) \in \mathfrak{H}_{D_{\nu+\rho+1}}$, such that for $z \in D_{\nu+\rho-1}$

$$|f_{M,\rho}(z) - f_{M,\rho-1}(z)| < \varepsilon_\rho. \tag{1.84}$$

Here $f_{M,0}(z) = f_M(z)$, $\rho = 1, 2, \cdots$. Hence it is easy to see that for $z \in D_k$ (k is fixed and $k > \nu$) and $\rho_1 > \rho_2 > k - n$ we have

$$|f_{M,\rho_1}(z) - f_{M,\rho_2}(z)| \leqslant \sum_{s=\rho_2+1}^{s=\rho_1} |f_{M,s}(z) - f_{M,s-1}(z)| < \sum_{s=\rho_2+1}^{s=\rho_1} \varepsilon_s.$$

Applying Cauchy's criterion for convergence we conclude that the sequence of functions $\{f_{M,\rho}(z)\}$ converges uniformly in the domain D to a certain function $f(z) \in \mathfrak{H}_D$. In addition, it follows from the inequality (1.84) that for $z \in D_\nu$,

$$|f(z) - f_M(z)| \leqslant \sum_{\rho=1}^\infty |f_{M,\rho}(z) - f_{M,\rho-1}(z)| < \sum_{\rho=1}^\infty \varepsilon_\rho < \frac{\delta}{2}. \tag{1.85}$$

Taking into account that $M \in D_\nu$ and $\mathfrak{D}_\nu \subset D_\nu$, we obtain from the inequalities (1.83) and (1.85)

$$|f(M)|+\frac{\delta}{2}>|f_M(M)|>\sup|f_M(\mathfrak{D}_\nu)|+\delta$$
$$>\sup|f(\mathfrak{D}_\nu)|-\frac{\delta}{2}+\delta=\sup|f(\mathfrak{D}_\nu)|+\frac{\delta}{2}$$

that is

$$|f(M)|>\sup|f(\mathfrak{D}_\nu)|.$$

This means that the domain D_ν is holomorphically convex relative to the domain D. Hence, in view of Lemma 1, our assertion follows.

LEMMA 3. *Let a bounded domain $D \subset C^n$ be approximated from the inside by a principal sequence of domains of holomorphy $\{D_\nu\}$. Then there exists a principal sequence $\{E_p\}$ approximating the domain D from the inside and possessing the property that each domain E_p is holomorphically convex relative to the domain E_{p+1}.*

PROOF. We denote by $d_{\mu\nu}(Q)$ the distance (understood in the sense stated in §11, Chapter II, (I)) from a point $Q \in \partial D_\mu$ to the boundary ∂D_ν. Further, let

$$M_{\mu\nu}=\max_{Q\in\partial D_\mu} d_{\mu\nu}(Q), \qquad m_{\mu\nu}=\min_{Q\in\partial D_\mu} d_{\mu\nu}(Q).$$

Analogously we denote by M_μ and m_μ, respectively, the maximal and minimal distances of points of the boundary ∂D_μ from the boundary ∂D of the domain D.

From the sequence of domains $\{D_\nu\}$ we choose a subsequence of domains D_{ν_p} in the following manner: 1) $D_{\nu_1} = D_1$; 2) the domain D_{ν_2} is so chosen that $M_{\nu_2} < m_{\nu_1}$; 3) the domain D_{ν_3} is so chosen that $M_{\nu_2\nu_3} < m_{\nu_1\nu_3}$, $M_{\nu_3} < m_{\nu_2}$ and so forth. In general the domain $D_{\nu_{p+1}}$, for $p > 1$, is so chosen that $M_{\nu_{p+1}} < m_{\nu_p}$ and

$$M_{\nu_p\nu_{p+1}} < m_{\nu_{p-1}\nu_{p+1}}. \tag{1.86}$$

All the domains D_ν, and in particular the domain $D_{\nu_{p+1}}$, are domains of holomorphy and thus are holomorphically convex. Therefore in view of the inequality (1.86) and Theorem 11.1, (I) we find that

$$\hat{D}_{\nu_{p-2},\nu_{p+1}} \Subset D_{\nu_p},$$

where $\hat{D}_{\nu_{p-1},\nu_{p+1}}$ is the convex hull of the domain $D_{\nu_{p-1}}$ relative to the family of functions holomorphic in the domain $D_{\nu_{p+1}}$. Reasoning further in the same way as in the proof of Theorem 2.1, we can construct the Weil analytic polyhedron E_p with defining function $f_j(z)$ holomorphic in the domain $D_{\nu_{p+1}}$, which satisfies

the condition

$$D_{\nu_{p-1}} \Subset E_p \Subset D_{\nu_p}.$$

Evidently $\lim_{p\to\infty} E_p = D.$[1)]

From Lemma 9 of §2 it follows that every function holomorphic in the domain E_p can be represented in this domain by a uniformly convergent series which consists of functions holomorphic in the domain D_{ν_p+1}, and accordingly also in the domain E_{p+1} so long as $E_{p+1} \Subset D_{\nu_p+1}$. Hence, taking into account the fact that E_p is a domain of holomorphy (see, for example, the corollary of Theorem 5.7), by Theorem 2.1 it follows that the domain E_p is holomorphically convex relative to the domain E_{p+1}. The lemma is proved.

Theorem 6.2 follows from the lemmas just proved.

LEMMA 4. *If a sequence of domains of holomorphy $\{D_\nu\}$ converges from inside to a bounded domain $D \subset C^n$, then for every number $\delta > 0$ one can find a domain of holomorphy $E \Subset D$, for which the maximal boundary distance in the domain D is less δ.*

EXPLANATION. By the *maximal boundary distance of the domain E in the domain D* we mean the maximum of the boundary distances of points $Q \in E$ in the domain D.

PROOF. Let $\mathfrak{D}$ be a domain for which $D_\delta \subset \mathfrak{D} \Subset D$. Here, as usual, $D_\delta = \{z \in D,\ d_D(z) > \delta\}$. We denote by δ_1 the minimal boundary distance of the domain $\mathfrak{D}$ in the domain D. Evidently[2)] $\delta_1 \le \delta$. From the sequence $\{D_p\}$ we single out a domain D_{p_0} with the maximal boundary distance less than $\delta_1/4$ in the domain D. It is clear that $\mathfrak{D} \Subset D_{p_0}$. Further, let $M = \{z \in D_{p_0},\ d_{D_{p_0}}(z) = \delta_1/2\}$. Evidently $M \subset D$ and $M \cap \overline{\mathfrak{D}} = \emptyset$, and in the domain D_p the minimal boundary distance of the set M is less than that of the domain $\mathfrak{D}$. Therefore, since D_{p_0} is a domain of holomorphy, and in view of Theorem 11.1, (I), to each point $Q \in M$ there corresponds a function $f_Q(z) \in \mathfrak{H}_{D_{p_0}}$ such that

$$\sup |f_Q(S(Q, r_Q))| > 1 > \sup |f_Q(\mathfrak{D})|. \tag{1.87}$$

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1) These arguments are not essentially different from those that were used for the proof of the lemma to Theorem 3.5.

2) Generally speaking $\delta_1 \ne \delta$, because the set of points of the domain D with boundary distance greater than δ may be disconnected. In constructing the domain $\mathfrak{D}$ we have to include in it points of the domain D with boundary distance greater than δ.

Reasoning in the same way as for the proof of Theorem 2.1 (see §2.2), we can construct an analytic polyhedron $E \supset \mathfrak{D}$ not containing a point belonging to the set M. Thus the domain E satisfies all the requirements stated in the lemma. The lemma is proved.

PROOF OF THEOREM 6.1. Starting from the sequence of domains $\{D_p\}$ given above, we define domains E_p satisfying the conditions of Lemma 4. These domains are chosen in such a way that: 1) the number δ_p, being equal to the maximal boundary distance of the domain E_p in the domain D, is less than the minimal boundary distance of the domain E_{p-1} in the domain D and 2) $\lim_{p\to\infty} \delta_p = 0$. As a result we obtain the sequence of domains $\{E_p\}$ satisfying the assumptions of Theorem 6.1. Hence it follows that D is a domain of holomorphy. Theorem 6.1 is proved.

3. Problem of convergence of holomorphy hulls. From the propositions we have proved in the preceding subsection follows

LEMMA 5. *If in the space* C^n: 1) *a bounded domain* D *is approximated from the inside by a sequence of domains* D_ν, $\nu = 1, 2, \cdots$;

2) *all of these domains have single-sheeted holomorphy hulls*;

3) *there exist domains of holomorphy* H_ν *for which*

$$D_\nu \subset H_\nu \subset H(D), \quad \nu = 1, 2, \cdots,$$

then $\lim_{\nu\to\infty} H_\nu = H(D)$.

PROOF. We consider the domain $H = \lim_{\nu\to\infty} H_\nu$. This domain H has the following properties:

1) $D \subset H$;

2) every point $Q \in H$ belongs to all domains H_ν with the number ν larger than some number ν_Q;

3) H is the largest domain satisfying the conditions 1) and 2).

We denote by $\mathfrak{D}_k$ the intersection of all domains H_ν with number $\nu \geq k$. As is well known, such an intersection $\mathfrak{D}_k$ consists of domains of holomorphy. Evidently $\mathfrak{D}_k \subset \mathfrak{D}_{k+1}$ and $\lim_{k\to\infty} \mathfrak{D}_k = H$. Hence on the basis of Theorem 6.1 we conclude that H is a domain of holomorphy. Since $D \subset H$, we have $H(D) \subset H$.

On the other hand, we must have $H \subset H(D)$. Indeed, every point $Q \in H$ belongs to the domains H_ν, while the latter satisfy the condition $H_\nu \subset H(D)$.

Thus $H = \lim_{\nu\to\infty} H_\nu \supset H(D)$ and $H = \lim_{\nu\to\infty} H_\nu \subset H(D)$. Hence we have

$$\lim_{\nu\to\infty} H_\nu = H(D).$$

The lemma is proved.

Now we can obtain the

THEOREM 6.3. *If in the space C^n a bounded domain D is approximated from the inside by a sequence of domains $\{D_\nu\}$, and all these latter domains have single-sheeted holomorphy hulls, then* $\lim_{\nu\to\infty} H(D_\nu) = H(D)$.

In other words, *under the conditions stated the function $H(D)$ is semicontinuous from the inside.*

The proof of this theorem is reduced to the application of the previous lemma by taking the domains $H(D_\nu)$ in place of the domains H_ν.

Theorem 6.3 may be extended to the cases of unbounded multiple-sheeted domains.[1)]

THEOREM 6.4. *In the space C^n let a bounded domain D be approximated by a sequence of domains $\{D_\nu\}$ and let all these latter domains have single-sheeted hulls. Then*

$$\lim_{\nu\to\infty} H(D_\nu) = H(\lim D_\nu) = H(D) \tag{1.88}$$

if and only if the domain D has no complementary holomorphy hull.

In other words, *the absence of a complementary holomorphy hull for the domain D is a necessary and sufficient condition that the function $H(D)$ should be continuous.*

PROOF. 1. SUFFICIENCY. Consider the intersections: $\mathfrak{D}_\nu = D_\nu \cap D$ and $E_\nu = H(D_\nu) \cap H(D)$. Evidently E_ν is a domain of holomorphy. Further we see that $\mathfrak{D}_\nu \subset D \subset H(D)$, $\mathfrak{D}_\nu \subset D_\nu \subset H(D_\nu)$, and therefore

$$\mathfrak{D}_\nu \subset H(D) \cap H(D_\nu) = E_\nu.$$

Consequently $\mathfrak{D}_\nu \subset E_\nu \subset H(D)$. In addition $\lim_{\nu\to\infty} \mathfrak{D}_\nu = D$, and hence, because of Lemma 5, there follows $\lim_{\nu\to\infty} E_\nu = H(D)$.

Hence it follows that each point $Q \in H(D)$ belongs to the domains $H(D_\nu)$ with number ν larger than some number ν_Q. We must still show that there does not exist a domain $R \supset H(D)$ every point of which is also contained in the infinite set of domains $H(D_\nu)$. For this purpose we consider a certain principal sequence of domains $\{B_\mu\}$ which approximates the domain D from the outside. Then

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1) See Behnke-Stein [2], (I). Many of the proofs in this article could be essentially simplified by the use of a chordal distance.

$H(B_\mu) \Supset H(D)$, which follows from the corollary of Theorem 13.4, (I) and the condition of single-sheetedness of the holomorphy hull $H(D)$. Since the domain D has no complementary holomorphy hull, $\lim_{\mu\to\infty} H(B_\mu) = H(D)$ (see the end of subsection 1 of the present section). It is evident that for each number μ one can find a number $\tilde{\nu}$ such that for $\nu > \tilde{\nu}$ we have the inclusion relation: $D_\nu \Subset B_\mu$, and there therefore $H(D_\nu) \Subset H(B_\mu)$. Hence it follows that every point belonging to all the domains $H(D_\nu)$ with sufficiently large number ν will belong to all the domains $H(B_\mu)$ with sufficiently large number μ, and since $\lim_{\mu\to\infty} H(B_\mu) = H(D)$, they also belong to the domain $H(D)$. With this the sufficiency of the condition indicated in Theorem 6.4 is proved.

2. NECESSITY. Suppose that $N(D) \neq H(D)$. Then it is easy to construct a sequence of domains $\{D_\nu\}$, $\lim_{\nu\to\infty} D_\nu = D$, for which in general $\lim_{\nu\to\infty} H(D_\nu)$ does not exist. To this end it suffices to take the domain D_ν in such a way that $D_{2m} \subset D$ and $\lim_{m\to\infty} D_{2m} = D$ (then $\lim_{m\to\infty} H(D_{2m}) = H(D)$), and the domains D_{2m+1} to form a principal sequence converging to the domain D from the outside (then $\lim_{m\to\infty} H(D_{2m+1}) = N(D)$). It is evident that in this case $\lim_{\nu\to\infty} H(D_\nu)$ does not exist. Then the necessity of the condition indicated in Theorem 6.4 is proved.

It is easy to construct examples of sequences of domains $\{D_\nu\}$, $\lim_{\nu\to\infty} D_\nu = D$, for which $H(D) \subset \lim_{\nu\to\infty} H(D_\nu) \subset N(D)$. In general, if $H(D) \neq N(D)$, then $\lim_{\nu\to\infty} H(D_\nu)$ may coincide with an arbitrary domain of holomorphy which contains the domain $H(D)$ and is included in the domain $N(D)$ (see Behnke-Stein [1]).

CHAPTER II

FUNDAMENTAL PROBLEMS. COHERENT ANALYTIC SHEAVES

§7. FORMULATION OF FUNDAMENTAL PROBLEMS. SOLUTION OF COUSIN'S FIRST PROBLEM FOR DOMAINS OF HOLOMORPHY OF THE SPACE C^n. APPLICATIONS

1. **Formulation of Cousin's first problem.** Let Z be a complex manifold, $\mathfrak{O}(Z) = \{\mathfrak{O}_z, \Pi', z \in Z\}$ be a sheaf of germs of holomorphic functions and $\mathfrak{M}^1(Z) = \{\mathfrak{M}^1_z, \Pi'', z \in Z\}$ be a sheaf of germs of meromorphic functions over the manifold Z. Here $\mathfrak{M}^1_z$ and $\mathfrak{O}_z$ are, respectively, the collections of germs of holomorphic and meromorphic functions at the point $z \in Z$, considered as abelian groups with respect to addition of germs entering into them, while the group $\mathfrak{O}_z$ is a subgroup of the group $\mathfrak{M}^1_z$. Consider the factor group $\mathfrak{H}_z = \mathfrak{M}^1_z/\mathfrak{O}_z$; an element of this factor group $\mathfrak{h}_z$, namely an equivalence class to which a germ $m_z \in \mathfrak{M}^1_z$ belongs, is called a *principal part* of that germ. Thus germs $m_z^{(1)}, m_z^{(2)} \in \mathfrak{h}_z$ (namely they have the identical principal part) if they are *equivalent with respect to subtraction*, i.e., if $m_z^{(2)} - m_z^{(1)} \in \mathfrak{O}_z$.

The sheaf $\mathfrak{O}(Z)$ is a subsheaf of the sheaf $\mathfrak{M}^1(Z)$, and the projection Π' is the restriction of the projection Π'', given on the space M of the sheaf $\mathfrak{M}^1(Z)$, to the space O of the sheaf $\mathfrak{O}(Z)$. We consider the factor sheaf:

$$\mathfrak{H}(Z) = \mathfrak{M}^1(Z)/\mathfrak{O}(Z) = \{\mathfrak{H}_z, \Pi, z \in Z\}.$$

It is called a *sheaf of principal parts of germs of meromorphic functions.* Its space H consists of elements $\mathfrak{h}_z$. A basis of the topology of this space is composed of a system of neighborhoods $V_{\mathfrak{h}_{z_0}} \subset H$ which are made up in the following manner: let $U_{z_0} \subset Z$ be a neighborhood of a point $z_0 \in Z$; then $V_{\mathfrak{h}_{z_0}}$ is the set of those elements $\mathfrak{H}_z$ which, at points $z \in U_{z_0}$, are principal parts of one and the same meromorphic function defined in the neighborhood U_{z_0} and belonging to

a germ of $\mathfrak{h}_{z_0}$. A section of the sheaf $\mathfrak{H}(Z)$ over some set $V \subset Z$ is called a system of *principal parts given on that set* V. We can now formulate the

FIRST (ADDITIVE) PROBLEM OF COUSIN. *A system of principal parts of germs of meromorphic functions is given on a set* Z. *It is required to define on the set* Z *a meromorphic function whose germs have the given principal parts at all points of that set.*

The question, whether this problem is solvable or not, can be formulated as follows: *is the projection* $\pi^* : \mathfrak{M}^1_Z \longrightarrow \mathfrak{H}_Z$ (or, expressed differently, $\pi^*: H^0(Z, \mathfrak{M}^1) \longrightarrow H^0(Z, \mathfrak{H})$, see Theorem 0.2 in Introduction) *generated by the homomorphism of sheaves* $\pi: \mathfrak{M}^1(Z) \longrightarrow \mathfrak{H}(Z)$ *an epimorphism?*

REMARK. In the sequel, if any first problem of Cousin is solvable on some set, we simply say that Cousin's first problem is solvable on it (in distinction from the case when a certain definite first problem of Cousin is solvable on it).

Similar remarks are also applicable to other fundamental problems considered here.

2. Formulation of Cousin's second problem. We denote by $\mathfrak{M}^2_z$ the collection of germs of meromorphic functions at a point $z \in Z$ (excluding the identically zero function), considered as an abelian group with respect to multiplication, and by $\mathfrak{F}_z$ the collection of germs of holomorphic functions invertible at the point z, namely, divisors of unity (see §4.2, Chapter I, (I)). These latter germs are represented at the point z by nonvanishing functions at that point and their collection $\mathfrak{F}_z$ is itself a group with respect to multiplication. In addition the group $\mathfrak{F}_z$ is a subgroup of the group $\mathfrak{M}^2_z$. We consider the sheaf $\mathfrak{F}(Z) = \{\mathfrak{F}_z, \pi', z \in Z\}$ which is a subsheaf of the sheaf $\mathfrak{M}^2(Z) = \{\mathfrak{M}^2_z, \pi'', z \in Z\}$.

At each point $z \in Z$ we form the factor group $\mathfrak{D}_z = \mathfrak{M}^2_z/\mathfrak{F}_z$. Elements $\mathfrak{D}_z$ of this factor group are called *germs of divisors of meromorphic functions*. If a germ $m_z \in \mathfrak{M}^2_z$ belongs to an equivalence class $\mathfrak{D}_z$, then $\mathfrak{D}_z$ is called the *divisor* of this germ m_z. Thus germs $m_z^{(1)}, m_z^{(2)} \in \mathfrak{D}_z$ if they are *equivalent with respect to division*, i.e., if $m_z^{(2)}/m_z^{(1)} \in \mathfrak{F}_z$.

If at least one germ $m_z \in \mathfrak{D}_z$ is a germ of a holomorphic function, then this is true for all germs $m_z \in \mathfrak{D}_z$. In this case $\mathfrak{D}_z$ is a germ of a *positive divisor.*

Since the sheaf $\mathfrak{F}(Z)$ is a subsheaf of the sheaf $\mathfrak{M}^2(Z)$, we may consider the factor sheaf

$$\mathfrak{D}(Z) = \mathfrak{M}^2(Z)/\mathfrak{F}(Z) = \{\mathfrak{D}_z, \ \pi, \ z \in Z\}.$$

It is called a *sheaf of germs of divisors* over the manifold Z. A section of this sheaf over some manifold $V \subset Z$ is called a *divisor over that manifold* V. This divisor is *positive* if it consists entirely of germs of positive divisors.

Now we can formulate the

SECOND (MULTIPLICATIVE) PROBLEM OF COUSIN. *There is given a divisor over a manifold* Z. *It is required to define on the manifold* Z *a meromorphic function whose germs have the given divisor at all points of that manifold.*

Here again the question of the solvability of this problem can be formulated as follows: *is the projection* $\pi^*: \mathfrak{M}_Z^2 \to \mathfrak{D}_Z$ (or, expressed differently, $\pi^*: H^0(Z, \mathfrak{M}^2) \to H^0(Z, \mathfrak{D})$) *generated by the homomorphism of sheaves* $\pi: \mathfrak{M}^2(Z) \to \mathfrak{D}(Z)$ *an epimorphism?*

3. Poincaré's problem. Remarks on the solution of fundamental problems. Cousin's problems are closely connected with the *problem of Poincaré. There is given a meromorphic function on a manifold* Z. *It is required to represent it there as a quotient of two holomorphic functions.*

In the strengthened problem of Poincaré it is required, in addition, that these latter functions should not have common holomorphic divisors at any point $z \in Z$.

Above we have considered solutions of Cousin's problems in several special cases. In §25, Chapter V, (I) we have proved Cousin's theorems. The first of these establishes that Cousin's first problem is solvable if the manifold Z coincides with the space C^n, or it is a polycylindrical domain of this space. The second theorem establishes that Cousin's second problem for positive divisors is solvable if the manifold Z coincides with the whole space C^n, or it is a polycylindrical domain $T_1 \times \cdots \times T_n$ (where all domains T_k, except possibly one, are simply-connected).

In §2.3, Chapter I we have proved that Cousin's first problem is solvable for the interior of the Weil polynomial polyhedron.

In obtaining these results we have not used the theory of sheaves. In essence, we shall also avoid use of it in §7, the first section of the present chapter, in the solution of Cousin's first problem for domains of holomorphy of the space C^n, though we shall begin with the deduction of a more general formulation of this problem. The theory of sheaves turns out to be essential for the solution of Cousin's problems for complex manifolds of general type.

4. Solution of Cousin's first problem for domains of holomorphy of the space C^n. The results explained in §2 of the preceding chapter allow us to prove without difficulty the following fundamental theorem:

THEOREM 7.1 (Oka [1], (I)). *Cousin's first problem is solvable for an arbitrary domain of holomorphy of the space* C^n.

PROOF. Let $D \subset C^n$ be a given domain of holomorphy. Using the property of holomorphic convexity and reasoning in the same way as for the proof of Theorem 2.1 (see §2.2 and §2.7, Chapter I), we construct a principal sequence of Weil polyhedra $\{\Delta_p, p = 1, 2, \cdots\}$ approximating the domain D from the inside. Here the defining functions $f_j^{(p)}(z)$, $j = 1, \cdots, N_p$, of each polyhedron Δ_p belong to the class of functions holomorphic in the domain D.

With each domain $\Delta_p \subset C_z^n$ we put into correspondence the set $\Sigma_p = \{z \in \Delta_p, w_j = f_j^{(p)}(z), j = 1, \cdots, N_p\}$ in the space $C_{z,w}^{n+N_p}$ of the variables $z_1, \cdots \cdots, z_n, w_1, \cdots, w_{N_p}$. As we have proved above (see Lemma 8 in §2, Chapter I), $\pi_1(\Sigma_p) = \Sigma_p$. Hence it follows that there exists a set $\tilde{\Sigma}_p \in \pi_0$ (this means that $\tilde{\Sigma}_p$ is a Weil polynomial set, see §2. 3–4, Chapter I) constituting an arbitrarily close neighborhood of the set Σ_p.

In the domain D Cousin's first problem is set up, i.e., a system of principal parts is given. We associate with the principal parts given at points $z \in \bar{\Delta}_p$ the corresponding points $(z, w) \in \bar{\Sigma}_p$. As a result Cousin's first problem is set up on the closed set $\bar{\Sigma}_p$, and accordingly on its (sufficiently close) neighborhood $\tilde{\Sigma}_p \in \pi_0$ as well. In view of Lemma 3 in §2, Chapter I, Cousin' s first problem is solvable on the closed set $\bar{\Sigma}$ lying inside the Weil polynomial set $\tilde{\Sigma}_p$. Suppose that a meromorphic function $G_p(z_1, \cdots, z_n, w_1, \cdots, w_{N_p})$ is its solution.

After the substitutions $w_j = f_j^{(p)}(z)$, $j = 1, \cdots, N_p$, we obtain the function $F_p(z) = G_p(z_1, \cdots, z_n, f_1^{(p)}(z), \cdots, f_{N_p}^{(p)}(z))$ which is the solution of the original Cousin's first problem for the closed domain $\bar{\Delta}_p$.

Now, after having constructed the function $F_p(z)$, we make use of the previous process to complete the proof of Theorem 25.1, (I) (Cousin's first theorem). As a result there will be constructed a function giving the solution of Cousin's first problem for the domain D. The proof of Theorem 7.1 is thus completed.

For the space C^2 it turns out that the inverse proposition is also true, and indeed this was proved by H. Cartan.[1)]

THEOREM 7.2. *The domain* $D \subset C_{w,z}^2$ *for which Cousin's first problem can be solved is necessarily a domain of holomorphy.*

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1) See H. Cartan [3]. Theorem 7.2 is not extended to the cases of the space C_z^n for $n \geq 3$.

PROOF. Suppose that Cousin's first problem is solvable in the domain D, but that D is not a domain of holomorphy. Then the domain D differs from its holomorphy hull $H(D)$ and there exists at least one boundary point P of the domain D such that it is the center of a hyperball K in which all functions holomorphic in the domain D are still holomorphic. Let Q be a point interior to the domains K and D. We take the segment PQ and find on it the boundary point P' of the domain D nearest to Q (evidently it belongs to K). We take an analytic plane passing through the straight line PQ as the plane $w = 0$ and the point P' as the origin of coordinates. We denote by D_0 the section of the domain D by the plane $w = 0$. Then P' is a boundary point of the set D_0. Let $z_1, z_2, \cdots, z_p, \cdots, \cdots, \lim_{p\to\infty} z_p = 0$, be some sequence of values z belonging to the section D_0. We consider a certain function $g(z)$ having the following properties: 1) it is holomorphic at all points of the z-plane and $g(z_p) = 0$; 2) the point $z = 0$ is an essentially singular point.

In the domain D we set up Cousin's first problem; we assign principal parts $\mathfrak{h}_P$ at a point $P \in D$ as follows:

$$\mathfrak{h}_P \ni \begin{cases} \dfrac{g(z)}{w}, & \text{if } \; w = 0 \text{ at the point } P, \\ 0, & \text{if } \; w \neq 0 \text{ at the point } P. \end{cases} \tag{2.1}$$

Evidently, for such a choice of principal parts $\mathfrak{h}_P$ the conditions for the formulation of Cousin's first problem will be satisfied. Suppose that a function $F(w, z)$, mermorphic in the domain D, as its solution. We form the function

$$G(w, z) = wF(w, z). \tag{2.2}$$

The functions $F(w, z)$ and $G(w, z)$ are clearly holomorphic everywhere in the domain D except at points on the plane $w = 0$. In view of formula (2.1), in some sufficiently small neighborhood V_P of the point $P(0, z) \in D$, we have

$$G(w, z) = w\left[\frac{g(z)}{w} + r_P(w, z)\right] = g(z) + wr_P(w, z). \tag{2.3}$$

Here $r_P(w, z)$ is some function holomorphic in the neighborhood V_P. Formula (2.2) does not define the function $G(w, z)$ at the point $P(0, z)$ but formula (2.3) provides the definition of the function $G(w, z)$ at that point. As a result the function $G(w, z)$ turns out to be holomorphic in the whole domain D. Thus it must be holomorphic in the hyperball K as well. But this is impossible, since $G(0, z) = g(z)$ and therefore the point $(0, 0)$ is essentially singular for the function $G(w, z)$. The theorem is proved.

PROOF OF HEFER'S THEOREM. The possibility of obtaining Weil's integral representation for analytic polyhedra is based on the following proposition.

THEOREM 7.3. *Let D_z be a domain of holomorphy in the space C^n_z of the variables $z_1, \cdots, z_n$, and let D_ζ be the same domain in the space C^n_ζ of the variables $\zeta_1, \cdots, \zeta_n$. Then to each function $Z(z)$ holomorphic in the domain D_z there correspond in the domain $D_\zeta \times D_z \subset C^{2n}_{\zeta,z}$ holomorphic functions $P_i(\zeta, z)$, $i = 1, \cdots, n$, such that*

$$Z(\zeta) - Z(z) = \sum_{i=1}^{n} (\zeta_i - z_i) P_i(\zeta, z). \tag{2.4}$$

This proposition has already been formulated in Theorem 22.1 in the first part of the present book. If the function $Z(z)$ is a polynomial or a rational function holomorphic in the domain D, the proposition is obvious, and it is easily verified in the case when the function $Z(z)$ is represented in a domain $\tilde{D}$ (a certain neighborhood of the domain D) as the limit of a uniformly convergent sequence of polynomials or rational functions holomorphic in the domain $\tilde{D}$. Theorem 7.3 in its general form was proved by Hefer [1], (I) in 1942.

The proof of Theorem 7.3 is reduced to the proof of two lemmas.

LEMMA 1. *Suppose that a domain of holomorphy $D \subset C^n_z$ and an $(n-1)$-complex-dimensional analytic plane E^{n-1} have a nonempty intersection $G = D \cap E^{n-1}$; let ϕ be a holomorphic function given on the set G. Then in the domain D there exists a holomorphic function coinciding with the given function ϕ on the set G.*

REMARK. If the intersection $G = D \cap E^{n-1}$ is split into certain connected components G_j, then we have to understand by the function ϕ the collection of holomorphic functions ϕ_j, each being given on the corresponding component G_j.

PROOF OF LEMMA 1. We can always assume that (if necessary after a suitable change of coordinate system) the equation of the plane E^{n-1} has the form: $z_1 = 0$. Then ϕ represents a holomorphic function of the variables $z_2, \cdots, z_n$.

Set up Cousin's first problem in the domain D; we assign principal parts $\mathfrak{H}_z$ as follows:

$$\mathfrak{h}_z \ni \begin{cases} \dfrac{\varphi(z_2, \ldots, z_n)}{z_1} & \text{for} \quad z \in G; \\ 0 & \text{for} \quad z \notin G. \end{cases}$$

It is clear that in this way a section of the sheaf $\mathfrak{H}(D)$ is given and a system of principal parts is defined over the domain D. By Oka's Theorem 7.1 there exists

in the domain D a holomorphic function $F(z)$ which is the solution of this problem of Cousin.

Consider the function $z_1F(z)$. It is defined and holomorphic at $z \in D\setminus G$; every point $z^{(0)} \in G$ has a neighborhood $U \subset D$ such that at $z \in U\setminus G$

$$\begin{gathered} F(z)-\frac{\varphi(z_2,\ldots,z_n)}{z_1}=h(z), \\ z_1F(z)=\varphi(z_2,\ldots,z_n)+z_1h(z), \end{gathered} \tag{2.5}$$

where $h(z)$ is a function holomorphic in the entire neighborhood U. The latter equation of (2.5) shows that the function $z_1F(z)$ is analytically continuable to the set G. As a result of this proposition we obtain there the function $\phi(z_2,\cdots,z_n)$ (since $z_1h(z)=0$ on the set G).

Thus, in the domain D we have defined the holomorphic function

$$\Phi(z)=\begin{cases} z_1F(z) & \text{for} \quad z\notin G, \\ \varphi(z_2,\ldots,z_n) & \text{for} \quad z\in G, \end{cases}$$

which satisfies all the conditions of the lemma. Hence the lemma is proved.

LEMMA 2. *Suppose that a domain of holomorphy $D\subset C_z^n$ and an $(n-k)$-complex-dimensional analytic plane $E^{n-k}=\{z_1=\cdots=z_k=0\}$, where $1\le k\le n$, have a nonempty intersection $G=D\cap E^{n-k}$; let $f(z)$ be a holomorphic function in the domain D which vanishes on the set G. Then in the domain D the representation*

$$f(z)=\sum_{i=1}^{k} z_iQ_i(z)$$

holds. Here $Q_i(z)$ are holomorphic functions in the domain D.

PROOF. We shall prove the lemma by induction. The assertion of the lemma is evident if the section of the domain D is produced by the plane E^{n-1}. Indeed, if for example $E^{n-1}=\{z_1=0\}$, we put $Q_1(z)=f(z)/z_1$; this function satisfies the requirements of the lemma. Suppose that the assertion of the lemma is true for the section of the domain D by the plane $E^{n-k+1}=\{z_1=\cdots=z_{k-1}=0\}$.

Consider the intersection $\mathfrak{D}$ of the domain D with the analytic plane $E_k^{n-1}=\{z_k=0\}$. Since the domain D is a domain of holomorphy, all connected components $\mathfrak{D}_j$ $(j=1,\cdots,N)$ of the domain $\mathfrak{D}$ are also domains of holomorphy in the space C^{n-1} of the variables $z_1,\cdots,z_{k-1},z_{k+1},\cdots,z_n$. In fact, along with the domain D all the domains $\mathfrak{D}_j\subset C^{n-1}$ are holomorphically convex, and hence they evidently possess the desired property.

The given function $f(z)$ is holomorphic at points of the set $\mathfrak{D}$ and equal to

zero on the intersection $\mathfrak{D} \cap \{z_1 = \cdots = z_{k-1} = 0\} = D \cap E^{n-k}$. Therefore, by assumption, on the set $\mathfrak{D}$ we have

$$f = \sum_{i=1}^{k-1} z_i Q_i^* (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n). \tag{2.6}$$

Here Q_i^* are holomorphic functions on the set $\mathfrak{D}$ (if the set $\mathfrak{D}$ consists of certain connected components $\mathfrak{D}_j$, the function Q_i^* are composed of holomorphic functions defined in each domain $\mathfrak{D}_j$ separately). By Lemma 1, with each function Q_i^* in the domain D we can associate a holomorphic function $Q_i(z_1, \cdots, z_n)$ coinciding with the function Q_i^* on the set $\mathfrak{D}$. In the domain D we consider the holomorphic function

$$\varphi(z) = f(z) - \sum_{i=1}^{k-1} z_i Q_i(z). \tag{2.7}$$

In view of relation (2.6) and the property of the functions $Q_i(z)$ it vanishes on the intersection $D \cap \{z_k = 0\}$. We have already noted that the lemma to be proved is correct for the section of the domain D by the plane E^{n-1}. Take as this plane the plane $z_k = 0$; we establish that in the domain D we have

$$\varphi(z) = z_k Q_k(z).$$

Substituting this expression for the function $\phi(z)$ into the relation (2.7) we obtain the required result. The lemma is proved.

COMPLETION OF THE PROOF OF THEOREM 7.3. By our conditions $Z(z)$ is a function holomorphic in the domain of holomorphy D_z. The difference $Z(\zeta) - Z(z)$ is holomorphic in the domain $D_\zeta \times D_z$ which is also a domain of holomorphy in the space $C^{2n}_{\zeta,z}$. This difference vanishes on the plane $E^n = \{z_i - \zeta_i = 0, i = 1, \cdots, n\}$. We put $z_i^* = \zeta_i - z_i$, $z_{n+i}^* = z_i$, $i = 1, \cdots, n$, and then apply Lemma 2 to the difference. As a result we obtain

$$Z(\zeta) - Z(z) = \sum_{i=1}^{n} z_i^* Q_i(z^*) = \sum_{i=1}^{n} (\zeta_i - z_i) P_i(\zeta, z),$$

where $P_i(\zeta, z) = Q_i(z^*)$. Theorem 7.3 is proved.

§8. COHERENT ANALYTIC SHEAVES OVER COMPLEX MANIFOLDS

1. Analytic sheaves. Let Z be a complex manifold, $V \subset Z$ an arbitrary open set, $\mathfrak{O}(Z) = \{\mathfrak{O}_z, z \in Z\}$ the sheaf of rings of germs of holomorphic functions, $\mathfrak{F}(Z) = \{\mathfrak{F}_z, z \in Z\}$ a certain sheaf of abelian groups over the manifold Z, $\mathfrak{O}(V)$

and $\mathfrak{F}(V)$ restrictions of these sheaves over the open set V, and let $\mathfrak{O}_V$ and $\mathfrak{F}_V$ be, respectively, the ring of sections and the group of sections of corresponding sheaves over the open set V. In other words, $\mathfrak{O}_V$ is the ring of functions holomorphic on the open set V.

DEFINITION (analytic sheaf).[1] The sheaf $\mathfrak{F}(Z)$ is said to be analytic if:

1) at every point $z \in Z$ the stalk $\mathfrak{F}_z$ possesses the structure of a $\mathfrak{O}_z$-module;

2) for every open set $V \in Z$ and every function $f \in \mathfrak{O}_V$ the mapping f^*: $\mathfrak{F}(V) \longrightarrow \mathfrak{F}(V)$, defined as the multiplication of each element $\phi_z \in \mathfrak{F}_z$ by the germ $f_z \in \mathfrak{O}_z$ for all $z \in V$, is continuous.

EXAMPLES. I. Consider the sheaf $\mathfrak{O}^q(Z)$, that is, the direct sum of q copies of the sheaf $\mathfrak{O}(Z)$. Here q is some positive integer. The stalk $\mathfrak{O}^q_z$ of this sheaf at a point $z \in Z$ consists of the system $(f^1_z, \cdots, f^q_z)$. These systems may also be represented as q-dimensional vectors with components $f^s_z \in \mathfrak{O}_z$, $s = 1, \cdots, q$. To each pair of elements $g \in \mathfrak{O}_z$ and $(f^1_z, \cdots, f^q_z) \in \mathfrak{O}^q_z$ there corresponds an element $(gf^1_z, \cdots, gf^q_z) \in \mathfrak{O}^q_z$. Thus the stalk $\mathfrak{O}^q_z$ has the structure of a $\mathfrak{O}_z$-module and accordingly the sheaf $\mathfrak{O}^q(Z)$ satisfies the first requirement of the definition of an analytic sheaf. Evidently the second requirement is also met by this sheaf. Thus the sheaf $\mathfrak{O}^q(Z)$ is analytic.

II. For each open set $V \in Z$ we consider an abelian group Ω^m_V of exterior differential forms of degree m, holomorphic on that set V (see §14.1, Chapter III, (I)). The collection of these groups defines the sheaf $\Omega^m(Z)$ of groups of germs of exterior differential holomorphic forms of degree m over the manifold Z. The group Ω^m_V is obviously an $\mathfrak{O}_V$-module. The stalk Ω^m_z of the sheaf $\Omega^m(Z)$ at the point $z \in Z$ is the inductive (direct) limit of the $\mathfrak{O}_V$-modules Ω^m_V relative to the filter of neighborhoods of the point z. Therefore the stalk Ω^m_z turns out to be an $\mathfrak{O}_z$-module. We have established that for the sheaf $\Omega^m(Z)$ the first requirement of the definition of an analytic sheaf is satisfied. Fulfillment of the second requirement is again obvious. Thus $\Omega^m(Z)$ is an analytic sheaf.

Suppose that at each point $z \in Z$ a submodule $\mathfrak{Q}_z$ is singled out in the $\mathfrak{O}_z$-module $\mathfrak{O}^q_z$. We shall say that *the collection of these submodules satisfies the condition* (Π), if for any open set $V \subset Z$, every element $f \in \mathfrak{O}^q_V$, representing the

1) The theory of analytic sheaves was constructed in H. Cartan's seminar. In our exposition we follow Cartan [5], Reports XI, XV, XVI, XVIII, and XIX.

germ $f_{z_1} \in \mathfrak{O}_{z_1}$ at some point $z_1 \in V$, represents the germ $f_{z_2} \in \mathfrak{O}_{z_2}$ at any other point z_2 of some neighborhood of the point z_1.

Fulfillment of the condition (П) is evidently a necessary and sufficient condition, that the union $\{\mathfrak{O}_z,\ z \in Z\}$ should represent an analytic subsheaf of the sheaf $\mathfrak{O}^q(Z)$.

EXAMPLES. I. Let M be a submodule of the $\mathfrak{O}_z$-module $\mathfrak{O}^q_Z$. At each point $z \in Z$ this module generates in the $\mathfrak{O}_z$-module $\mathfrak{O}^q_z$ a submodule M_z.[1] The collection of these submodules $\{M_z,\ z \in Z\}$ satisfies the condition (П) and accordingly it is an analytic subsheaf of the sheaf $\mathfrak{O}^q(Z)$. The sheaf is said to be *generated by the module M*.

II. Suppose that elements $f^1, \cdots, f^p \in \mathfrak{O}^q_Z = H^0(Z, \mathfrak{O}^q)$. At each point $z \in Z$ we single out from the ring $\mathfrak{O}_z$ all systems of elements $(c^1_z, \cdots, c^p_z)$ for which $c^1_z f^1_z + \cdots + c^p_z f^p_z = 0$. Here $f^i_z \in \mathfrak{O}^q_z$ is the germ represented by the element f^i at the point z. Together, these systems $(c^1_z, \cdots, c^p_z) \in \mathfrak{O}^p_z$ evidently constitute a submodule of the $\mathfrak{O}_z$-module $\mathfrak{O}^p_z$. This submodule is called a *module of relations among the elements* $f^1, \cdots, f^p$ *at the point* $z \in Z$ and is denoted by the symbol $R_z(f^1_z, \cdots, f^p_z)$, or more concisely by $R_z(f^1, \cdots, f^p)$. The collection of the modules $R_z(f^1, \cdots, f^p)$ satisfies the condition (П) and accordingly is an analytic subsheaf of the sheaf $\mathfrak{O}^p(Z)$. This sheaf is called a *sheaf of relations* among the elements $f^1, \cdots, f^p$ and is denoted by the symbol $R(f^1, \cdots, f^p)$.

DEFINITION (analytic homomorphism of analytic sheaves). Let $\mathfrak{F}(Z)$ and $\mathfrak{G}(Z)$ be analytic sheaves. A homomorphism of the sheaves ϕ: $\mathfrak{F}(Z) \longrightarrow \mathfrak{G}(Z)$ is said to be analytic if at every point $z \in Z$ the homomorphism $\phi_z: \mathfrak{F}_z \longrightarrow \mathfrak{G}_z$ induced by the homomorphism ϕ is permissible [2] in the structure of $\mathfrak{O}_z$-modules. If an analytic homomorphism turns out to be a monomorphism or an epimorphism, then it is called correspondingly an *analytic monomorphism* or *epimorphism*. If an analytic homomorphism is simultaneously monomorphic and epimorphic, then it is called an *analytic isomorphism*.

Suppose that an analytic sheaf $\mathfrak{F}(Z)$ is a subsheaf of an analytic sheaf $\mathfrak{G}(Z)$.

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1) Let A be a Λ-module (where Λ is a ring with a unit), and let elements $u_1, u_2, \cdots \in A$. It is said that a system of elements $\{u_1, u_2, \cdots\}$ generates a submodule B *in the* Λ-module A, if an element $u \in A$ is represented in the form of a finite form $\lambda_1 u_1 + \lambda_2 u_2 + \cdots \cdots$, where $\lambda_1, \lambda_2, \cdots \in \Lambda$, if and only if $u \in B$.

2) A homomorphism $\psi: A \to B$, where A and B are Λ-modules and Λ is a ring with a unit, is said to be *permissible* if $\psi(\lambda a) = \lambda\psi(a)$ for $a \in A$ and $\lambda \in \Lambda$.

If at every point $z \in Z$ the group $\mathfrak{F}_z$ is a submodule of the $\mathfrak{O}_z$-module $\mathfrak{G}_z$, then $\mathfrak{F}(Z)$ is said to be *analytic sheaf in the induced structure*.

A factor sheaf $\mathfrak{G}(Z)$ of an analytic sheaf $\mathfrak{F}(Z)$ is an *analytic sheaf in the factor-structure*, if $\mathfrak{G}(Z)$ is an analytic sheaf and at every point $z \in Z$ the group $\mathfrak{G}_z$ is a factor module of the $\mathfrak{O}_z$-module $\mathfrak{F}_z$ in the factor-structure.[1)]

THEOREM 8.1. *Let ϕ: $\mathfrak{F}(Z) \longrightarrow \mathfrak{G}(Z)$ be an analytic homomorphism of analytic sheaves, and let $\mathfrak{R}(Z)$ (a subsheaf of the sheaf $\mathfrak{F}(Z)$) be the kernel, $\mathfrak{J}(Z)$ (a subsheaf of the sheaf $\mathfrak{G}(Z)$) be the image, and $\mathfrak{S}(Z)$ (a factor sheaf $\mathfrak{G}(Z)/\mathfrak{J}(Z)$) be the cokernel, respectively, of the homomorphism ϕ. Then the sheaves $\mathfrak{R}(Z)$ and $\mathfrak{J}(Z)$ are analytic sheaves in the induced structure and the sheaf $\mathfrak{S}(Z)$ is an analytic sheaf in the factor-structure.*

The assertion of this theorem follows immediately from the definition of an analytic homomorphism.

2. A property of the ring $\mathfrak{O}_z$. Let $\mathfrak{O}_n$ be the ring of germs of functions holomorphic at the origin of coordinates of the space C^n and let $\Phi_1, \cdots, \Phi_\rho$ be a finite system of elements from the group $\mathfrak{O}_n^q$. We denote by $M(\Phi_1, \cdots, \Phi_\rho)$ the submodule of the $\mathfrak{O}_n$-module $\mathfrak{O}_n^q$ generated by the elements $\Phi_1, \cdots, \Phi_\rho$.

In the stalk $\mathfrak{O}_n^q$ we introduce the concept of a uniform convergence: we shall say that a sequence of elements $F_\nu \in \mathfrak{O}_n^q$, $\nu = 1, 2, \cdots$, $F_\nu = (f_{\nu 1}, \cdots, f_{\nu q})$, where all $f_{\nu r} \in \mathfrak{O}_n$, $r = 1, \cdots, q$, converges uniformly to an element $F \in \mathfrak{O}_n^q$, where $F = (f_1, \cdots, f_q)$ and all $f_s \in \mathfrak{O}_n$, if there exists a neighborhood U of the origin of coordinates, and functions $\tilde{f}_{\nu s}$, $\tilde{f}_s$ in U which represent respective germs, such that $\lim_{\nu\to\infty} \tilde{f}_{\nu s} = \tilde{f}_s$ and this limit is approached uniformly in the neighborhood of U.

THEOREM 8.2 (H. Cartan [5], XI). *For any numbers q, n and any elements $\Phi_1, \cdots, \Phi_\rho \in \mathfrak{O}_n^q$ the submodule $M(\Phi_1, \cdots, \Phi_\rho)$ is closed in the stalk $\mathfrak{O}_n^q$* (in the sense of the above definition of uniform convergence).

PROOF. A collection of linear continuous mappings λ_i: $\mathfrak{O}_n^q \longrightarrow \mathfrak{O}_n$, $i = 1, \cdots, \rho$, will be called a *construction for the system* $(\Phi_1, \cdots, \Phi_\rho)$ if for every

1) Let Λ be a ring with a unit, let A be a Λ-module, and let B be a submodule of A. Then on the factor group A/B the structure of a Λ-module (*factor-structure*) is defined there in the following way: if $\alpha \in A/B$ and $a \in A$ is a representative of the coset defined by the element α in the group A, then an element $\lambda\alpha$ (where $\lambda \in \Lambda$) is a coset to which the element $\lambda a \in A$ belongs. This definition is independent of the choice of the element a, since if $(a_1 - a_2) \in B$, then also $\lambda a_1 - \lambda a_2 = \lambda(a_1 - a_2) \in B$.

element $f \in M(\Phi_1, \cdots, \Phi_\rho)$ the equality $F = \Sigma_{i=1}^{\rho} \lambda_i(F)\Phi_i$ holds. It is easy to see that the assertion of Theorem 8.2 is true for the system $(\Phi_1, \cdots, \Phi_\rho)$ if there is a construction for this system. Therefore Theorem 8.2 follows from the following lemma.

LEMMA 1. *For any numbers q, n and any elements $\Phi_1, \cdots, \Phi_\rho \in \mathfrak{O}_n^q$ there exists a construction for the system $(\Phi_1, \cdots, \Phi_\rho)$.*

PROOF OF LEMMA 1. We shall carry out the proof by double induction on n and q. For $n = 0$ and $q = 1$ the construction may be formed without difficulty, since then the stalk $\mathfrak{O}_n^q$ coincides with the field of complex numbers and all of $\Phi_1, \cdots, \Phi_\rho$ are complex numbers. If $\Phi_1 = \cdots = \Phi_\rho = 0$, then we may take as the construction an arbitrary system of mappings; if one of these quantities (say Φ_1 to be definite) is different from zero, then we may take as the construction the collection of mappings: $\lambda_1(z) = z/\Phi_1$, $\lambda_i(z) = 0$, $i = 2, \cdots, \rho$. Here z is an arbitrary complex number.

Now the proof of the lemma is reduced to two inductive processes that are made possible by the following facts:

I. If the lemma is true for the stalk $\mathfrak{O}_n^{q-1}$, then it is also true for the stalk $\mathfrak{O}_n^q$, where $n \geq 0$ and $q \geq 2$.

II. If the lemma is true for the stalk $\mathfrak{O}_{n-1}^q$, then it is also true for the stalk $\mathfrak{O}_n^1 = \mathfrak{O}_n$, where $n \geq 1$ and $q \geq 1$.

PROOF OF I. Suppose that elements $\Phi_1, \cdots, \Phi_\rho \in \mathfrak{O}_n^q$ and that ϕ_i is the first component of the element Φ_i. Then by the assumption of the induction there exists a construction for the system $(\Phi_1, \cdots, \Phi_\rho)$; let λ_i: $\mathfrak{O}_n^1 \longrightarrow \mathfrak{O}_n$ be this construction. For each element $F \in \mathfrak{O}_n^q$ we consider the element $F - \Sigma_{i=1}^{\rho} \lambda_i(f)\Phi_i$, where f is the first component of the element F. If $F \in M(\Phi_1, \cdots, \Phi_\rho)$, then the first component of this difference is equal to zero and the latter can be considered as an element of the stalk $\mathfrak{O}_n^{q-1}$. Let $\psi_1, \cdots, \psi_\rho \in \mathfrak{O}_n^{q-1}$ be elements corresponding to the elements $\Phi_1, \cdots, \Phi_\rho \in M(\Phi_1, \cdots, \Phi_\rho)$ and let $\mu_1, \cdots, \mu_\rho$ be a construction for the system $(\psi_1, \cdots, \psi_\rho)$. Then, for every element $F \in M(\Phi_1, \cdots, \Phi_\rho)$, the equality

$$F = \sum_{i=1}^{\rho} \lambda_i(f)\Phi_i + \sum_{i=1}^{\rho} \mu_i\left(F - \sum_{j=1}^{\rho} \lambda_j(f)\Phi_j\right)\psi_i$$

holds. Putting $\psi_i = \Phi_i - \Sigma_{i=1}^{\rho} \lambda_i(\phi_j)\,\Phi_j$ and then reducing similar terms, we obtain the equality

$$F = \sum_{i=1}^{\rho} \nu_i(F)\Phi_i$$

for every $F \in M(\Phi_1, \cdots, \Phi_\rho)$. Thus the transformations $\nu_1, \cdots, \nu_\rho$ constitute the desired construction. The assertion I is proved.

PROOF OF II. Consider elements $\Phi_1, \cdots, \Phi_\rho \in \mathfrak{O}_n$; if $\Phi_1 = \cdots = \Phi_\rho = 0$, then the mappings $\lambda_i = 0$, $i = 1, \cdots, \rho$, form a construction of these elements ($\lambda = 0$ is the mapping which carries all the elements of the ring $\mathfrak{O}_n$ into the zero element). Suppose that one of these elements, for example, $\Phi_1 \not\equiv 0$. By the preparation theorem of Weierstrass (Theorem 4.2, (I)) one can find, possibly after a linear transformation of variables, a distinguished pseudopolynomial P, arranged in powers of the variable z_n (its coefficients are holomorphic functions of the variables $z_1, \cdots, z_{n-1}$), such that $P = \Phi_1 \mu$. Here μ is an invertible holomorphic function in some neighborhood of the origin of coordinates.

By a theorem on power series,[1] for any holomorphic functions B and P one can find a pseudopolynomial H arranged in powers of the variable z_n and a holomorphic function Q (here the functions B, P, Q and the coefficients of the pseudopolynomial H are assumed to be holomorphic in some neighborhood of the origin of coordinates) such that

$$B = P \cdot Q + H.$$

In addition, if $P = \sum_{k=0}^{\infty} P_k(z_1, \cdots, z_{n-1})\, z_n^k$ and $P_k(0, \cdots, 0) = 0$ for $k = 0, 1, \cdots, s-1$, while $P_s(0, \cdots, 0) \neq 0$, then the degree of the pseudopolynomial is equal to $s - 1$. The functions Q and H here are uniquely defined by the functions B and P; fixing the function P they depend linearly on the function B.

We apply this theorem to the case when P is the pseudopolynomial introduced above. We find that for every element $f \in \mathfrak{O}_n$ there exists a holomorphic function $Q(f)$ and a pseudopolynomial $R(f)$ of degree $p - 1$, where p is the degree of the pseudopolynomial P, such that

$$f = P \cdot Q(f) + R(f).$$

(Here $Q(f)$ and $R(f)$ depend linearly on the element f.) In particular we have

$$\Phi_i = P \cdot Q(\Phi_i) + R(\Phi_i) = P \cdot Q_i + R_i.$$

Pseudopolynomials R_i may be considered as elements of the stalk $\mathfrak{O}_{n-1}^{p-1}$ (so

1) See, for example, Bochner and Martin [1], (I), Chapter IX, §1, Lemma 1.

long as they are defined by their coefficients). By the hypothesis of the induction there exist constructions $\lambda_2, \cdots, \lambda_\rho$ for the elements $R_2, R_3, \cdots, R_\rho \in \mathfrak{O}^{p-1}_{n-1}$. We associate with each element $f \in \mathfrak{O}_n$ the sum

$$\mu Q(f)\,\Phi_1 + \sum_{i=2}^{\rho} \lambda_i(R(f))\,R_i.$$

If in the sum we replace the pseudopolynomials R_i by $\Phi_i - \mu Q_i \Phi_1$, we obtain a linear combination of the form $\Sigma_{i=1}^{\rho} \nu_i(f)\,\Phi_i$. The transformations $\nu_1, \cdots, \nu_\rho$ form the desired construction.

Thus Lemma 1 and therewith Theorem 8.2 are proved.

3. **Coherent subsheaf of the sheaf $\mathfrak{O}^q(Z)$.**

DEFINITION. An analytic subsheaf $\mathfrak{F}(Z)$ of the sheaf $\mathfrak{O}^q(Z)$ is said to be *coherent at a point* $z \in Z$ if there exists a neighborhood U of the point z and a system of elements $f^i \in \mathfrak{O}^q_U$, $i = 1, \cdots, p$, with the following property: at every point $\zeta \in U$ the stalk $\mathfrak{F}_\zeta$ coincides with a submodule of the $\mathfrak{O}_\zeta$-module $\mathfrak{O}^q_\zeta$ generated by elements f^i_ζ. Here $f^i_\zeta \in \mathfrak{O}^q_\zeta$ (where $\zeta \in U$) is a germ represented at the point $\zeta \in U$ by the vector f^i.

The subsheaf $\mathfrak{F}(Z)$ is said to be *coherent* if it is coherent at all points $z \in Z$.

From this definition one can easily conclude that:

If the subsheaf $\mathfrak{F}(Z)$ is coherent at a point $z \in Z$, then it is coherent at all points of some neighborhood V of that point.

If the sheaf $\mathfrak{F}(Z)$ is coherent at a point $z \in Z$, V is a certain neighborhood of that point z, and $f^{*(i)} \in \mathfrak{O}^q_V$, $i = 1, \cdots, p^*$, is a finite system generating in the $\mathfrak{O}_z$-module $\mathfrak{O}^q_z$ the submodule $\mathfrak{F}_z$, then the system $f^{*(1)}, \cdots, f^{*(p^*)}$ generates in the $\mathfrak{O}_\zeta$-module $\mathfrak{O}^q_\zeta$ the submodule $\mathfrak{F}_\zeta$ at all points ζ in some neighborhood of the point z.

In fact, in view of the definition of a coherent sheaf, at all points $\zeta \in U$ the system of elements $f^i \in \mathfrak{O}^q_U$, $i = 1, \cdots, p$, generates in the $\mathfrak{O}_\zeta$-module $\mathfrak{O}^q_\zeta$ the submodule $\mathfrak{F}_\zeta$. In particular this situation occurs at $\zeta = z$, and therefore we have

$$f^{*(j)}_z = \sum_{i=1}^{p} a^j_i f^i_z, \qquad j = 1, \ldots, p^*, \tag{2.8}$$

where $a^j_i \in \mathfrak{O}_z$. On the other hand, by assumption,

$$f^i_z = \sum_{j=1}^{p^*} b^i_j f^{*(j)}_z, \qquad i = 1, \ldots, p, \tag{2.9}$$

where $b^i_j \in \mathfrak{O}_z$. Equations (2.8) and (2.9) are satisfied in some neighborhood W of

the point z; consequently at every point $\zeta \in W$ the systems $\{f^i\}$ and $\{f^{*(i)}\}$ generate one and the same submodule of the $\mathfrak{O}_\zeta$-module $\mathfrak{O}^q_\zeta$. But the system $\{f^i\}$ generates at every point $\zeta \in W \subset U$ the submodule $\mathfrak{F}_\zeta$ (by the definition of a coherent subsheaf); therefore the system $\{f^{*(i)}\}$ also generates the submodule $\mathfrak{F}_\zeta$ in this neighborhood. From what has been proved follows the

THEOREM 8.3. *If analytic subsheaves* $\mathfrak{F}(Z)$ *and* $\mathfrak{G}(Z)$ *of the sheaf* $\mathfrak{O}^q(Z)$ *are coherent at a point* $z \in Z$ *and* $\mathfrak{F}_z = \mathfrak{G}_z$, *then* $\mathfrak{F}_\zeta = \mathfrak{G}_\zeta$ *in some neighborhood of the point* z.

From the definition of a coherent subsheaf there follows the

THEOREM 8.4. *If* $\mathfrak{F}(Z)$ *is an analytic coherent subsheaf of the sheaf* $\mathfrak{O}^q(Z)$, $\mathfrak{G}(Z)$ *is an analytic subsheaf of the sheaf* $\mathfrak{O}^q(Z)$, *and there exists an analytic epimorphism of the sheaf* $\mathfrak{F}(Z)$ *onto the sheaf* $\mathfrak{G}(Z)$, *then the sheaf* $\mathfrak{G}(Z)$ *is coherent.*

Moreover the following theorem is valid:

THEOREM 8.5. *For the coherency of an analytic subsheaf* $\mathfrak{F}(Z)$ *of the sheaf* $\mathfrak{O}^q(Z)$ *it is necessary and sufficient that at every point* $z \in Z$ *there should exist*: 1) *an open set* $U \subset Z$ *containing the point* z; 2) *a submodule* $M \subset \mathfrak{O}^q_U$ *in the structure of an* $\mathfrak{O}_U$*-module such that at all points* $\zeta \in U$ *the submodule of the* $\mathfrak{O}_\zeta$*-module* $\mathfrak{O}^q_\zeta$, *generated by elements from the image of the module* M *in the stalk* $\mathfrak{O}^q_\zeta$, *is the stalk* $\mathfrak{F}_\zeta$.

PROOF. The necessity results from the definition of a coherent subsheaf, while the sufficiency follows from the fact that the submodule $M \subset \mathfrak{O}^q_U$ as an $\mathfrak{O}_U$-module has a finite set of generators.

In particular, if M is a submodule of the $\mathfrak{O}_Z$-module $\mathfrak{O}^q_Z$, at each point $z \in Z$ it generates a submodule of the $\mathfrak{O}_z$-module $\mathfrak{O}^q_z$. As a result we obtain an analytic sheaf $\mathfrak{M}(Z)$ that is coherent by Theorem 8.5.

4. The Oka-Cartan theorem. The following fundamental theorem was first proved by Oka [6], (I). In the terminology of the theory of sheaves it was formulated and proved by H. Cartan [5], XV.

THEOREM 8.6 (Oka-Cartan). *For any finite system of elements* $f_1, \cdots, f_p$ *from* $\mathfrak{O}^q_{Z^n} = H^0(Z^n, \mathfrak{O}^q)$ *the sheaf of relations* $R(f_1, \cdots, f_p)$ *is coherent.*

We shall prove this theorem by double induction on the numbers q and n. We denote the assertion of the theorem for given numbers n and q by $\mathrm{OC}(n, q)$. Evidently the assertion $\mathrm{OC}(0, q)$ is correct. Therefore it is sufficient to prove:

I. *Let* $n > 0$ *and* $q > 1$; *if* $\mathrm{OC}(n, q')$ *is true for all* $q' < q$, *then so is* $\mathrm{OC}(n, q)$.

II. *Let* $n \geq 1$; *if* $\mathrm{OC}(n-1, q)$ *is true for all numbers* q, *then so is* $\mathrm{OC}(n, 1)$.

PROOF OF I. Let $(f_1, \cdots, f_p)$ be a system of elements from $\mathfrak{O}^q_{Z^n}$. Each of these elements f_i $(i = 1, \cdots, p)$ in turn represents a system of elements $f_{ij} \in \mathfrak{O}_{Z^n}$ $(j = 1, \cdots, q)$. Consider the elements $g_i \in \mathfrak{O}^{q-1}_{Z^n}$ with components f_{ij} $(j = 1, \cdots \cdots, q-1)$. Evidently the sheaf $R(f_1, \cdots, f_p)$ is a subsheaf of the sheaf $R(g_1, \cdots, g_p)$. The latter sheaf is coherent by the hypothesis of the induction.

Let a point $z \in Z^n$. Then there exist an open set $U \ni z$ and a finite system of elements $h_k \in \mathfrak{O}^p_U$ $(k = 1, \cdots, N)$ generating at any point $\zeta \in U$ the module $R_\zeta(g_1, \cdots, g_p)$. Every element of this module may be represented in the form $\Sigma_{k=1}^N a_k h_k^{(\zeta)}$, where $a_k \in \mathfrak{O}_\zeta$. We consider a submodule $\mathfrak{F}_\zeta \in \mathfrak{O}^N_\zeta$ consisting of those systems $\{a_1, \cdots, a_N\}$ for which

$$\sum_{k=1}^{N} a_k h_k^{(\zeta)} \in R_\zeta\ (f_1, \ldots, f_p). \tag{2.10}$$

Their collection $\mathfrak{F}(U)$ satisfies the condition (II) in subsection 1 of the present section and accordingly it is an analytic subsheaf of the sheaf $\mathfrak{O}^N(U)$. The mapping $(a_1, \cdots, a_N) \longrightarrow \Sigma_{k=1}^N a_k h_k$ defines an epimorphism of the sheaf $\mathfrak{F}(U)$ onto the restriction of the sheaf $R(f_1, \cdots, f_p)$ over the set U. In view of Theorem 8.4 our assertion will be proved if we establish the coherency of the sheaf $\mathfrak{F}(U)$.

The inclusion relation (2.10) means that

$$\sum_{i=1}^{p}\left(\sum_{k=1}^{N} a_k h_{ki}\right) f_{iq} = \sum_{k=1}^{N} a_k \left(\sum_{i=1}^{p} h_{ki} f_{iq}\right) = 0.$$

Thus $\mathfrak{F}(U)$ is a sheaf of relations among the functions $\Sigma_{i=1}^p h_{ki} f_{iq}$, but since the assertion $\mathrm{OC}(n, 1)$ is correct, it is coherent.

PROOF OF II. Since Theorem 8.6 is of local character and Z^n is a complex manifold, it suffices to prove the theorem for the case $Z^n = C^n$. Consider a certain system of functions $f_1, \cdots, f_p$ holomorphic in the neighborhood of the origin of coordinates of the space C^n. By the preparation theorem of Weierstrass (Theorem 4.2, (I)) these functions may be replaced (after a linear substitution of the coordinates $z_1, \cdots, z_n$,[1] and up to holomorphic factors distinct from zero at the origin of coordinates) by pseudopolynomials $\psi_1, \cdots, \psi_p$. The latter will be arranged in powers of the variable $\mathfrak{z} = z_n$; their coefficients will be holomorphic functions of the variables $z_1, \cdots, z_{n-1}$ in the neighborhood of the origin of

1) See Theorem 4.3, in which a single function f was concerned. However, this theorem also remains valid for a system of functions.

coordinates; the leading coefficient will be equal to 1. Evidently our assertion II will be proved if we establish that the *sheaf of relations* $R(\psi_1, \cdots, \psi_p)$ *is coherent at the origin of coordinates of the space* C^n *of the variables* $z_1, \cdots, z_{n-1}, \mathfrak{z}$.

We prove the following lemma:

LEMMA 2. *The module of relations* $R_z(\psi_1, \cdots, \psi_p)$ *at every point* $z = (z_1, \cdots, z_{n-1}, \mathfrak{z})$, *in which the pseudopolynomials* $\psi_1, \cdots, \psi_p$ *are holomorphic, is generated by a system* $(c_1, \cdots, c_p)$ *consisting of pseudopolynomials in the variable* $\mathfrak{z}$ *of degree not higher than* α. *Here* α *is the maximal degree of the pseudopolynomials* $\psi_1, \cdots, \psi_p$.

We remark that the coherency of the sheaf $R(\psi_1, \cdots, \psi_p)$ at the origin of coordinates follows from Lemma 2. Indeed, the relations among ψ_i for which the coefficients c_i are pseudopolynomials of degree not higher than α form an analytic sheaf relative to the sheaf $\mathfrak{O}(C^{n-1})$, where C^{n-1} is the space of the variables $z_1, \cdots, z_{n-1}$. The last sheaf is coherent, since the assertion OC$(n-1, q)$ is correct by the hypothesis of the induction. Thus the proof of the assertion II is reduced to that of Lemma 2.

PROOF OF LEMMA 2. Suppose that the pseudopolynomial ψ_p has the degree α and that at a point $a \in C^n$ its coefficients are holomorphic. Then in the neighborhood of the point a the equality

$$\psi_p = \psi'\psi'' \tag{2.11}$$

holds, where ψ' is a distinguished pseudopolynomial in $\mathfrak{z} - \mathfrak{z}_0$ ($\mathfrak{z}_0$ being the coordinates $\mathfrak{z}$ of the point a), vanishing at the point a, while ψ'' is a polynomial (in $\mathfrak{z}$) of maximal degree, which is different from zero at the point a. The possibility of the representation (2.11) comes from the preparation theorem of Weierstrass. From this theorem it follows immediately that $\psi_p = \tilde{\psi}'\tilde{\psi}''$, where $\tilde{\psi}'$ is a distinguished pseudopolynomial and $\tilde{\psi}''(a) \neq 0$. We decompose the pseudopolynomial $\tilde{\psi}'$ into a product of irreducible (distinguished) pseudopolynomials and denote by ψ' the product of those of them which vanish at the point a, and by ψ'' the product of the remainders on $\tilde{\psi}''$. In view of Theorem 4.4, (I) ψ'' is a pseudopolynomial. Thus we obtain the decomposition (2.11). We denote by α' and α'' the degrees (in $\mathfrak{z}$) of the pseudopolynomials ψ' and ψ''. Evidently $\alpha' + \alpha'' = \alpha$.

Consider the relation among pseudopolynomials $\psi_1, \cdots, \psi_p$:

$$\sum_{i=1}^{p} c_i\psi_i = 0. \tag{2.12}$$

Here $c_1, \cdots, c_p$ are functions holomorphic at the point a. Applying the theorem for power series that has been used for the proof of Theorem 8.2, we obtain:

$$c_i = \psi' Q_i' + R_i = \psi_p\left(Q_i'\psi''^{-1}\right) + R_i = \psi_p Q_i + R_i$$

$$i = 1, \ldots, p-1,$$

where Q_i are functions holomorphic at the point a, and R_i are pseudopolynomials in $\mathfrak{z}$ of degree not higher than $\alpha' - 1$. Because of this equality the system $(c_1, \cdots, c_p)$ may be reduced to the linear combination

$$\begin{aligned}(c_1, \ldots, c_p) &= (\psi_p, 0, \ldots, 0, -\psi_1)Q_1 \\ &+ (0, \psi_p, \ldots, 0, -\psi_2)Q_2 + (0, \ldots, 0, \psi_p, -\psi_{p-1})Q_{p-1} \\ &+ (R_1, \ldots, R_p), \text{ where } R_p = c_p + \sum_{i=1}^{p-1}\psi_i Q_i.\end{aligned}$$

The systems $(\psi_p, 0, \cdots, 0, -\psi_1), \cdots, (0, \cdots, 0, \psi_p, -\psi_{p-1})$ belong to the sheaf of relations among pseudopolynomials (in the variable $\mathfrak{z}$) of degree not higher than α. It remains to consider the system $(R_1, \cdots, R_p)$. In view of relation (2.12) we have

$$\sum_{i=1}^{p-1} R_i\psi_i + (R_p\psi'')\psi' = 0.$$

Here $\Sigma_{i=1}^{p-1} R_i\psi_i$ is a pseudopolynomial of the variable $\mathfrak{z}$. Hence, by applying Theorem 4.4, (I), we conclude that $R_p\psi''$ is also a pseudopolynomial of the variable $\mathfrak{z}$. Since $\psi''(a) \neq 0$, the function ψ''^{-1} is holomorphic at the point a and

$$(R_1, \ldots, R_p) = \psi''^{-1}(\psi'' R_1, \ldots, \psi'' R_p).$$

We shall show that the degree of each of the pseudopolynomials $\psi'' R_i$, $i = 1, \cdots, p$, is not higher than α. For $i = 1, \cdots, p-1$ this follows from the equality $\alpha' + \alpha'' = \alpha$ (since the degree of each of the pseudopolynomials $R_1, \cdots, R_{p-1}$ is not higher than $\alpha' - 1$). From the equality

$$\sum_{i=1}^{p} (\psi'' R_i)\psi_i = 0$$

it is seen that the degree of the pseudopolynomial $\psi'' R_p\psi_p$ is not higher than 2α. But the degree of the pseudopolynomial ψ_p is equal to α and therefore the degree of the pseudopolynomial $\psi'' R_p$ cannot be higher than α.

Lemma 2 and therewith Theorem 8.6 are proved.

We consider two important consequences of the Oka-Cartan theorem.

THEOREM 8.7. *The intersection $\mathfrak{F}(Z) \cap \mathfrak{G}(Z)$ of two coherent analytic subsheaves $\mathfrak{F}(Z)$ and $\mathfrak{G}(Z)$ of the sheaf $\mathfrak{O}^q(Z)$ is again a coherent subsheaf.*

PROOF. Let U be a neighborhood of a point $z \in Z$, and let $\{f_i\}$ and $\{g_j\}$ be finite systems of elements of the group $\mathfrak{O}^q_U$ generating, respectively, modules $\mathfrak{F}_\zeta$ and $\mathfrak{G}_\zeta$ at every point $\zeta \in U$. On the open set U consider the sheaf of relations $R(f_i, g_j)$ of the system (f_i, g_j). It is coherent by Theorem 8.6. For each element $c = (c_{1i}, c_{2j}) \in R(f_i, g_j)$ we set $\phi(c) = \Sigma_i c_{1i} f_i = -\Sigma_j c_{2j} g_j$. As a result there is defined a homomorphism ϕ of the coherent sheaf $R(f_i, g_j)$ onto the sheaf $\mathfrak{F} \cap \mathfrak{G}$; this last sheaf turns out to be coherent by Theorem 8.4.

THEOREM 8.8. *Suppose that:* 1) *M is a submodule of the $\mathfrak{O}_Z$-module $\mathfrak{O}^q_Z$ and a point $z \in Z$;* 2) *a function $g \in \mathfrak{O}_Z$ has the property that the inclusion relations $f_z \in \mathfrak{O}^q_z$ and $g_z f_z \in M_z$ always imply the inclusion relation $f_z \in M_z$ (here M_z is a submodule of the $\mathfrak{O}_z$-module $\mathfrak{O}^q_z$ generated by the submodule M). Then there exists a neighborhood U of the point z such that at any point $\zeta \in U$ the inclusion relation $g_\zeta f_\zeta \in M_\zeta$ follows from the inclusion relation $f_\zeta \in M_\zeta$.*

PROOF. The sheaf $M(Z)$ defined by the module M is coherent by Theorem 8.5. Therefore, at an arbitrary point $z \in Z$, one can find a neighborhood V and a finite system of elements $m_k \in M$ such that at any point $\zeta \in V$ the stalk M_ζ is generated by the elements m_k.

The element $g \in \mathfrak{O}_V$ defines q elements of the group $\mathfrak{O}^q_V$

$$g_1 = (g, 0, \ldots, 0), \; g_2 = (0, g, \ldots, 0), \ldots, g_q = (0, 0, \ldots, g).$$

Consider elements $f_\zeta \in \mathfrak{O}^q_\zeta$ and $gf_\zeta = \Sigma_{i=1}^q f^i_\zeta g_i$ (here f^i_ζ are components of the element f_ζ).

The sheaf of relations among elements m_k and g_i is coherent in view of Theorem 8.6. Consequently there exists a neighborhood U' of the point z (where $U' \subset V$) and a finite system (p^k_s, h^i_s), $s = 1, 2, \cdots$, consisting of functions holomorphic in this neighborhood U' and generating at every point $\zeta \in U'$ the sheaf of relations $R_\zeta(m_k, g_i)$. For all indices s we have

$$\sum_k p^k_s m_k + \sum_i h^i_s g_i = 0.$$

Therefore $gh_s \in M_z$, where $h_s = (h^1_s, \cdots, h^q_s) \in \mathfrak{O}^q_z$. Hence, from the hypothesis of the theorem to be proved, it follows that $h_s \in M_z$ as well. Consequently $h_s \in M_\zeta$ at any point ζ of some neighborhood U of the point z. Let $f = (f^1, \cdots, f^q) \in \mathfrak{O}^q_Z$ and $gf_\zeta \in M_\zeta$. Then there exist elements $r^k_\zeta \in \mathfrak{O}_\zeta$ such that

$$\sum_k r_\zeta^k m_k + \sum_i f_\zeta^i g_i = 0.$$

The system (r_ζ^k, f_ζ^i) represents a linear combination with coefficients from the ring $\mathfrak{O}_\zeta$ of the systems (p_s^k, h_s^i), $s = 1, 2, \cdots$. Hence it follows that the element f_ζ may be represented as a linear combination of elements h_s with coefficients from the ring $\mathfrak{O}_\zeta$. Therefore, since $h_s \in M_\zeta$ $(s = 1, 2, \cdots)$, we have $f_\zeta \in M_\zeta$ as well, which was to be proved.

5. General concept of an analytic coherent sheaf. Let W be some subspace of a complex manifold Z, and let $\mathfrak{O}(W)$ be the restriction of the sheaf $\mathfrak{O}(Z)$ on this subspace, which defines thereon an induced complex structure (see §15.1, (I) and following, where we have considered similar structures for certain subsets of a complex space).

On the basis of the sheaf $\mathfrak{O}(W)$ we introduce the concepts of an analytic sheaf and a coherent analytic subsheaf of the sheaf $\mathfrak{O}^q(W)$ over the space W. When we pass from the manifold Z to the space W, Theorems 8.1, 8.3, and 8.4 do not undergo any change, but Theorem 8.5 must be replaced by the following proposition:

THEOREM 8.5$_1$. *For the coherency of an analytic subsheaf $\mathfrak{F}(W)$ of the sheaf $\mathfrak{O}^q(W)$ it is necessary and sufficient that each point $z \in W$ should have a neighborhood $U \subset Z$ in which the module $\mathfrak{O}^q_U$ generates the stalk of this sheaf $\mathfrak{F}_\zeta$ at all points $\zeta \in U \cap W$.*

The following property of a coherent subsheaf of the sheaf $\mathfrak{O}^q(W)$ lies at the root of the general definition of a coherent sheaf.

THEOREM 8.9. *For the coherency of an analytic subsheaf $\mathfrak{F}(W)$ of the sheaf $\mathfrak{O}^q(W)$ it is necessary and sufficient that each point $z \in W$ should have a neighborhood $U \subset W$ for which the restriction $\mathfrak{F}(U)$ of this sheaf is isomorphic to a factor sheaf $\mathfrak{O}^p(U)/R(U)$. Here $R(U)$ is a coherent subsheaf of the sheaf $\mathfrak{O}^p(U)$ and $p > 0$ is some integer.*

PROOF. Sufficiency of the stated condition follows from Theorem 8.4; we shall prove its necessity. By virtue of the coherency of the sheaf $\mathfrak{F}(W)$ one can find at every point $z \in W$ a neighborhood $U \subset W$ and elements $f^i \in \mathfrak{O}^q_U$, $i = 1, \cdots, p$, with the following property: at every point $\zeta \in U$ the submodule of the $\mathfrak{O}_\zeta$-module $\mathfrak{O}^q_\zeta$ generated by the elements f^i_ζ is the stalk $\mathfrak{F}_\zeta$. Let $R(f^1, \cdots, f^p)$ be the sheaf of relations among elements f^i over U. At each point $\zeta \in U$ it defines a submodule $R_\zeta(f^1, \cdots, f^p)$ of the module $\mathfrak{O}^p_\zeta$. By Theorem 8.6 this

sheaf is coherent; hence our proposition follows.

DEFINITION (coherent analytic sheaf). An analytic sheaf $\mathfrak{F}(W)$ is said to be *coherent* if each point $z \in W$ has a neighborhood $U \subset W$ such that the restriction of the sheaf $\mathfrak{F}(W)$ is isomorphic to a factor sheaf $\mathfrak{O}^p(U)/R(U)$ in the structure of an analytic sheaf $\mathfrak{O}^p(U)$. Here $R(U)$ is a coherent subsheaf of the sheaf $\mathfrak{O}^p(U)$ and $p > 0$ is some integer.

The sheaf of germs of (exterior) differential holomorphic forms serves as an example of a coherent analytic sheaf which is not a subsheaf of the sheaf $\mathfrak{O}^p(W)$.

THEOREM 8.10 (extension of the Oka-Cartan theorem to the case of an arbitrary coherent sheaf). *For any finite system $f^1, \cdots, f^p \in \mathfrak{F}_W = H^0(W, \mathfrak{F})$ of sections of an analytic coherent sheaf $\mathfrak{F}(W)$ over the space W the sheaf of relations $R(f^1, \cdots, f^p)$ is a coherent subsheaf of the sheaf $\mathfrak{O}^p(W)$.*

PROOF. In view of the coherency of the sheaf $\mathfrak{F}(W)$ there exists at each point $z \in W$ a neighborhood $U \subset W$ such that the sheaf $\mathfrak{F}(U)$ is isomorphic to a factor sheaf $\mathfrak{O}^q(U)/R(U)$, where q is a positive integer and $R(U)$ a coherent subsheaf of the sheaf $\mathfrak{O}^q(U)$. In other words, one can find q sections $\psi^j \in \mathfrak{F}_U$ generating the group $\mathfrak{F}_U$; $R(U)$ is the coherent sheaf of relations among ψ^j. By what has been said we have

$$f_z^i = \sum_j a_j^i \psi_z^j, \qquad i = 1, \ldots, p,$$

where a_j^i are functions holomorphic at the point z. We choose the neighborhood U sufficiently small so that the functions a_j^i remain holomorphic and at all points $\zeta \in U$ the equation

$$f_\zeta^i = \sum_j a_j^i \psi_\zeta^j, \qquad i = 1, \ldots, p \tag{2.13}$$

holds there.

Suppose that the relation

$$\sum_{i=1}^{p} c_\zeta^i f^i = 0 \tag{2.14}$$

holds, where $(c_\zeta^1, \cdots, c_\zeta^p) \in \mathfrak{O}_\zeta^p$ and $\zeta \in U$. Then, because of (2.13), the relation (2.14) may be rewritten as

$$\sum_j \Big(\sum_i c_\zeta^i a_j^i\Big) \psi_\zeta^j = 0.$$

Thus the system $\sum_{i=1}^p a_j^i c_\zeta^i$, $j = 1, 2, \cdots$, belongs to the module of relations $R_\zeta(\psi^1, \psi^2, \cdots)$. Since the sheaf of relations among elements ψ_ζ^j (where $\zeta \in U$)

is coherent, this module is generated by a finite system of mappings $\{b^k_j,\ k = 1, 2, \cdots\}$. Consequently there exists a finite system of functions $\gamma^k_\zeta \in \mathfrak{O}_\zeta$ such that

$$\sum_i c^i_\zeta a^i_j + \sum_k \gamma^k_\zeta b^k_j = 0, \qquad j = 1, 2, \ldots .$$

The sheaf of relations among elements c^i_ζ, γ^k_ζ is coherent by Theorem 8.6. From what has been said above it follows that the sheaf being studied is: 1) a factor sheaf of a coherent sheaf; 2) a subsheaf of the sheaf $\mathfrak{O}^p(W)$. Therefore it is coherent in view of Theorem 8.4.

From the theorem just proved there follows the

THEOREM 8.11. *For the coherency of an analytic subsheaf $\mathfrak{H}'(W)$ of an analytic coherent sheaf $\mathfrak{H}(W)$, it is necessary and sufficient that each point $z \in W$ should have a neighborhood U in which there exists a finite system of elements $h_i \in \mathfrak{H}_U$ generating the stalk $\mathfrak{H}'_\zeta$ at every point $\zeta \in U$.*

REMARK. If $\mathfrak{H}(W) = \mathfrak{O}^q(W)$, then Theorem 8.11 is reduced to the definition of a coherent subsheaf of the sheaf $\mathfrak{O}^q(W)$; it has been formulated in §8.3 for the case when $W = Z$. The second condition in the definition of a coherent sheaf given in the present subsection, namely, the condition of coherency of the sheaf R, is satisfied automatically for a subsheaf of a coherent sheaf (in particular the sheaf $\mathfrak{O}^q(W)$) and accordingly may be discarded (as was done in the definition stated in subsection 3); however, it is essential in the general definition.

The following theorem has many applications.

THEOREM 8.12. *Let $\mathfrak{F}(W)$ and $\mathfrak{G}(W)$ be coherent analytic sheaves, and let $\phi: \mathfrak{F} \longrightarrow \mathfrak{G}$ be their analytic homomorphism. Then the image* Im ϕ, *the kernel* Ker ϕ, *and the cokernel* Coker ϕ *are coherent analytic sheaves.*

PROOF. Let a point $z \in Z$. Then there exists a neighborhood $U \subset W$ of this point and p elements $f^i \in \mathfrak{F}_U$ generating at any point $\zeta \in U$ the stalk $\mathfrak{F}_\zeta$ as a $\mathfrak{F}_U$-module.

The image Im $\phi = \mathfrak{J}(W)$, as a subsheaf of the coherent sheaf $\mathfrak{H}(W)$, is coherent by Theorem 8.11.

The kernel Ker $\phi = \mathfrak{N}(W)$ is also a coherent sheaf. Indeed, we consider at each point $\zeta \in U$ those systems of p elements $c^i_\zeta \in \mathfrak{O}_\zeta$, $i = 1, \cdots, p$, for which

$$\sum_i c^i_\zeta \varphi(f^i_\zeta) = 0,$$

where $\phi(f^i_\zeta) \in \mathfrak{H}_\zeta$. In view of Theorem 8.10 these systems form a coherent sub-

sheaf of the sheaf $\mathfrak{O}^p(U)$. The mapping $\{c^i\} \to \Sigma_i c^i f^i$ of this sheaf into the sheaf $\mathfrak{F}(U)$ has the sheaf $\mathfrak{R}(U)$ as its image. As we have shown above, it is coherent. Hence our assertion follows.

The cokernel Coker $\phi = \mathfrak{S}(W)$ is the factor sheaf $\mathfrak{H}(W)/\mathfrak{J}(W)$. At each point $\zeta \in U$ the stalk $\mathfrak{J}_\zeta$ is generated by a finite system of elements $h'^{(i)} \in \mathfrak{H}_U$. In some neighborhood V of the point z there exists a finite system of q elements $h^i \in \mathfrak{H}_V$ generating the stalk $\mathfrak{H}_\zeta$ at each point $\zeta \in V$. We may assume that $V \subset U$. In view of Theorem 8.10 the sheaf R of relations among elements h^i_ζ is coherent; the sheaf $\mathfrak{H}(V)$ is isomorphic to the factor sheaf $\mathfrak{O}^q(V)/R$.

We choose, in any way, elements $h''^{(i)} \in \mathfrak{O}^q_V$, representing elements $h'^{(i)} \in \mathfrak{H}_V$ (the elements $h''^{(i)}$ belong to the corresponding equivalence classes; in order to choose them, it may be necessary to diminish the neighborhood V; see the remark at the end of the proof). They generate a coherent subsheaf $\tilde{R}$ of the sheaf $\mathfrak{O}^q(V)$. Evidently the subsheaf $R + \tilde{R}$ of the sheaf $\mathfrak{O}^q(V)$ is coherent and the sheaf $\mathfrak{S}(V)$ is isomorphic to the factor sheaf $\mathfrak{O}^q(V)/(R + \tilde{R})$. Hence the assertion follows.

REMARK. From the fact that for the homomorphism ϕ: $\mathfrak{A}(W) \to \mathfrak{B}(W)$ the mapping $A|_U \to B|_U$ is surjective (U is a neighborhood of the point $z \in W$, and A and B are spaces of corresponding sheaves) it does not follow that the image $\phi(\mathfrak{A}_U)$ coincides with the collection of the sections $\mathfrak{B}_U$ (see Theorem 0.3). However, if $b'^{(1)}, \cdots, b'^{(p)} \in \mathfrak{B}_U$ is some finite set of sections and $b_z'^{(1)}, \cdots, b_z'^{(p)} \in \mathfrak{B}_z$ are elements belonging to these sections in the stalk $\mathfrak{B}_z$, then in this case one can find elements $a_z^1, \cdots, a_z^p \in \mathfrak{A}_z$ such that $\phi(a_z^i) = b_z'^{(i)}$, $i = 1, \cdots, p$. For each value of the index i there exist a neighborhood V_i of the point z and an element $a'^{(i)} \in \mathfrak{A}_{V_i}$ representing the element a_z^i. Let $V = [\bigcap_i V_i] \cap U \subset U$, and let $b^i \in \mathfrak{B}_V$ be the image of the element $b'^{(i)} \in \mathfrak{B}_U$ and $a^i \in \mathfrak{A}_V$ be the image of the element $a'^{(i)} \in \mathfrak{A}_{V_i}$. Then $\phi_V(a^i) = b^i$.

6. Complex manifolds. Let $E = E^p$ be a complex manifold properly imbedded in a complex manifold $Z = Z^n$ (see §18.4, Chapter III, (I)). In this case we may always assume that each point $z \in Z$ (but not $z \in E$, as would be so for an analytic, locally regular imbedding) has a neighborhood $U \subset Z$ such that either $E \cap U$ is empty or $E \cap U = \{z_{p+1} = \cdots = z_n = 0\} \cap U$. Here $z_1, \cdots, z_n$ are suitably chosen complex local coordinates on Z within the domain U.

We denote by $\mathfrak{E}_z$ the ideal of the ring $\mathfrak{O}_z$ which consists, at the point $z \in E$, of all germs $f_z \in \mathfrak{O}_z$ that vanish on the manifold E within the limit of some

neighborhood of the point z (on E) and is equal to $\mathfrak{O}_z$ if the point $z \notin E$. The collection $\mathfrak{S}(Z) = \{\mathfrak{S}_z, z \in Z\}$ forms an analytic subsheaf of the sheaf $\mathfrak{O}(Z)$; it is called the *sheaf of manifold* E. A similar sheaf of ideals may be considered also in the more general case when E is an arbitrary submanifold of the manifold Z. We have the

THEOREM 8.13. *The sheaf* $\mathfrak{S}(Z)$ *of the submanifold* E *is coherent.*

PROOF. We shall show that the sheaf $\mathfrak{S}(Z)$ is coherent at any point $z \in Z$.

This is obvious if the point $z \notin E$, since then $\mathfrak{S}_z = \mathfrak{O}_z$. We must consider the case when the point $z \in E$. Then we may assume that $E \cap U = \{z_{p+1} = \cdots \cdots = z_n = 0\} \cap U$, where $U \subset Z$ is some neighborhood of z which is the origin of local coordinates $z_1, \cdots, z_n$. In the neighborhood U take a finite system of functions $z_{p+1}, \cdots, z_n \in \mathfrak{S}_U$. If the point $\zeta \in U \setminus E$, then at least one of these functions is distinct from zero at the point ζ; therefore the above system generates in the $\mathfrak{O}_\zeta$-module a submodule coinciding with the entire stalk $\mathfrak{O}_\zeta = \mathfrak{S}_\zeta$. If the point $\zeta \in U \cap E$, then any function belonging to the germ $f_\zeta \in \mathfrak{S}_\zeta$ may be (as is seen from its Taylor expansion) represented in some neighborhood of the point ζ in the form $z_{p+1}f_{p+1} + \cdots + z_nf_n$, where f_i, $i = p + 1, \cdots, n$, are holomorphic functions in that neighborhood of the point ζ. Thus a finite system of elements from the stalk $\mathfrak{S}_\zeta$ generates at each point $\zeta \in U$ a submodule of the $\mathfrak{O}_\zeta$-module $\mathfrak{O}_\zeta$, equal to $\mathfrak{S}_\zeta$. Consequently the sheaf $\mathfrak{S}(Z)$ is coherent at the point z. The theorem is proved.

On the complex submanifold E of the complex manifold Z we consider two structural sheaves: the sheaf ${}_E\mathfrak{O}(E)$ defined over E as well as over the complex manifold (in an intrinsic way), and the sheaf ${}_Z\mathfrak{O}(E)$ which is the restriction over the submanifold E of the structural sheaf $\mathfrak{O}(Z)$ of the manifold Z. Sections of these sheaves over the manifold E are, respectively, E-holomorphic and Z-holomorphic functions on that manifold.

The restriction on E of that sheaf $\mathfrak{S}(Z)$ serves for the kernel of an essential homomorphism ϕ: ${}_Z\mathfrak{O}(E) \longrightarrow {}_E\mathfrak{O}(E)$ (under which, to restrictions of holomorphic functions of the manifold Z on E, there correspond holomorphic functions of the manifold E coinciding with them on E). In other words, the equality: ${}_E\mathfrak{O}(E) = {}_Z\mathfrak{O}(E)/\mathfrak{S}(E)$ holds there.

Starting from this equality we can extend the sheaf ${}_E\mathfrak{O}(E)$ to the whole manifold Z by setting ${}_E\mathfrak{O}(Z) = {}_Z\mathfrak{O}(Z)/\mathfrak{S}(Z)$. At points $z \in Z \setminus E$ the stalk of this sheaf is a null group. The section of this sheaf over the manifold Z, the element

of the group $H^0(Z, {}_Z\mathfrak{O}/\mathfrak{S})$, is an E-holomorphic function on the manifold E.

Let W be a subspace of the manifold Z, and let $V = W \cap E$ be the corresponding subspace of the manifold E, and $\mathfrak{F}(V)$ be a sheaf over the space V. We must distinguish the concepts of a Z-analytic sheaf $\mathfrak{F}(V)$ (when $\mathfrak{F}_z$ at points $z \in V$ is a ${}_Z\mathfrak{O}_z$-module) and an E-analytic sheaf $\mathfrak{F}(V)$ (when $\mathfrak{F}_z$ at points $z \in V$ is a ${}_E\mathfrak{O}_z$-module). Evidently every E-analytic sheaf is a Z-analytic sheaf (if $\lambda \in {}_Z\mathfrak{O}_z$ and $f \in \mathfrak{F}_z$, then $\lambda f = \phi(\lambda) f$, where $\phi(\lambda) \in {}_E\mathfrak{O}_z$).

THEOREM 8.14. *Let $\mathfrak{F}(V)$ be some E-analytic coherent sheaf over the space $V = W \cap E$. Then the sheaf defined by the equation*

$$\tilde{\mathfrak{F}}_z = \begin{cases} \mathfrak{F}_z, & if \quad z \in V, \\ 0, & if \quad z \in W \setminus V, \end{cases}$$

is a Z-analytic coherent sheaf over the space W.

PROOF. From the fact that $\mathfrak{F}(V)$ is an E-analytic sheaf it obviously follows that $\tilde{\mathfrak{F}}(W)$ is a Z-analytic sheaf. Coherency of the sheaf $\tilde{\mathfrak{F}}(W)$ at points $z \in W \setminus V$ is evident. Consider a point $z \in V$. Then the sheaf $\mathfrak{F}(V \cap U_1)$, where $U_1 \subset W$ is some neighborhood of the point z, is isomorphic to the factor sheaf ${}_E\mathfrak{O}^q(V \cap U_1)/R(V \cap U_1)$. Here $q > 0$ is a certain integer and $R(V \cap U_1)$ is a coherent analytic subsheaf of the sheaf ${}_E\mathfrak{O}^q(V \cap U_1)$. Because of the coherency of the subsheaf $R(V \cap U_1)$ there exists a finite set of systems of q functions, E-holomorphic in a neighborhood $U_2 \subset W$ of the point z, which generate at any point $\zeta \in V \cap U_2$ a submodule R_ζ in the ${}_E\mathfrak{O}_\zeta$-module ${}_E\mathfrak{O}^q_\zeta$. From the charts of the atlas of the manifold Z, which contain the point z (where the local coordinate system is placed) we single out a neighborhood $\tilde{U} \subset Z$ of the point Z in which all functions $f(z_1, \cdots, z_p)$, E-holomorphic in the domain $U = \tilde{U} \cap E$, remain E-holomorphic. Then all the functions E-holomorphic in the neighborhood $U \subset W$ of the point z are necessarily Z-holomorphic in the neighborhood $\tilde{U} \subset Z$ of the point z. In addition we require that $U \subset U_2 \subset U_1$. Then each of the systems generating the submodule R_ζ at the point $\zeta \in U$ is the trace on U of a system of Z-holomorphic functions in the domain $\tilde{U}$. These systems generate in the sheaf ${}_Z\mathfrak{O}^q(W \cap \tilde{U})$ a coherent subsheaf $\tilde{R}(W \cap \tilde{U})$ which induces a subsheaf $R(V \cap U)$ of the sheaf ${}_E\mathfrak{O}(V \cap U)$. Consider the sheaf $\mathfrak{S}^q(W \cap \tilde{U})$ equal to the direct sum of q copies of the sheaf $\mathfrak{S}(W \cap \tilde{U})$. It is a subsheaf of the sheaf ${}_Z\mathfrak{O}^q(W \cap \tilde{U})$; its coherency follows from Theorem 8.13. Therefore the sheaf

$\mathfrak{G}^q(W \cap \widetilde{U}) + \widetilde{R}(W \cap \widetilde{U})$ is also a coherent subsheaf of the sheaf ${}_Z\mathfrak{O}^q(W \cap \widetilde{U})$. The sheaf induced by the factor sheaf ${}_Z\mathfrak{O}^q(W \cap \widetilde{U})/(\widetilde{R} + \mathfrak{G}^q)$ in the neighborhood $\widetilde{U}$ coincides with the sheaf induced in that neighborhood by the sheaf $\mathfrak{F}(W)$. Hence our assertion follows.

REMARK. Concerning the possibility of extending the results in this subsection to an arbitrary analytic set, see Thimm [1], (I) and Kuhlmann [1], (I).

§9. COHERENT ANALYTIC SHEAVES OVER COMPLEX MANIFOLDS HAVING PROPERTIES (A) AND (B)[1)]

1. Auxiliary proposition.

THEOREM 9.1. *Let K be a compactum in a manifold Z. For any coherent analytic sheaf $\mathfrak{F}(K)$ one can find a neighborhood $V \subset Z$ of the compactum K and an analytic coherent sheaf $\mathfrak{G}(V)$, such that its restriction over the compactum K turns out to be analytically isomorphic to the sheaf $\mathfrak{F}(K)$.*

We preface the proof of Theorem 9.1 with a series of lemmas.

LEMMA 1. *Let $\mathfrak{F}(E)$ and $\mathfrak{G}(E)$ be coherent analytic sheaves over a subspace E of a complex manifold Z, and let ϕ and ψ be analytic homomorphisms $\mathfrak{F}(E) \longrightarrow \mathfrak{G}(E)$. Then the set of those points $z \in E$ for which $\phi_z = \psi_z$ is open in E.*

PROOF. We replace the homomorphism ϕ by $\phi - \psi$ (for such a replacement there corresponds to an element $f_\zeta \in \mathfrak{F}_\zeta$ an element $[\phi_\zeta(f_\zeta) - \psi_\zeta(f_\zeta)] \in \mathfrak{G}_\zeta$ instead of an element $\phi_\zeta(f_\zeta) \in \mathfrak{G}_\zeta$; here the point $\zeta \in E$) and the homomorphism ψ by 0. As a result we reduce the lemma to the case when $\psi = 0$. Because of the coherency of the sheaf $\mathfrak{F}(E)$ over the neighborhood of the point $z \in E$ there exists a finite system $\{f_i\}$ of sections of this sheaf which generates an $\mathfrak{O}_\zeta$-module $\mathfrak{F}_\zeta$ at every point $\zeta \in E$ sufficiently close to the point z. Since ϕ is an analytic homomorphism, in order that $\phi = 0$ it is necessary and sufficient that $\phi(f_i) = 0$ for all i. However, it is evident that for each index i the set of those points ζ for which $\phi_\zeta(f_i) = 0$ is open. Hence our assertion follows.

LEMMA 2. *Let $\mathfrak{F}(Z)$ and $\mathfrak{G}(Z)$ be coherent analytic sheaves over a complex manifold Z and let ϕ_z: $\mathfrak{F}_z \longrightarrow \mathfrak{G}_z$ be a homomorphism of the group $\mathfrak{F}_z$ into the group $\mathfrak{G}_z$ permissible in the structure of the $\mathfrak{O}_z$-module, where z is some point of Z. Then there exists a neighborhood U of the point z and an analytic homo-*

1) The theory of coherent analytic sheaves over complex manifolds having properties (A) and (B) was constructed in H. Cartan's seminar. In our exposition we follow Cartan [5], Reports XVII, XVIII, XIX. See also Cartan [4], (I).

morphism $\psi\colon \mathfrak{F}(U) \longrightarrow \mathfrak{G}(U)$ *such that* $\psi_z = \phi_z$.

In other words, *the homomorphism of stalks of coherent analytic sheaves at a certain point* z, *permissible in the structure of the* $\mathfrak{O}_z$*-module, may be extended to an analytic homomorphism of these sheaves over some neighborhood of the point* z.

PROOF. Since the sheaves $\mathfrak{F}(Z)$ and $\mathfrak{G}(Z)$ are coherent, at any point $\zeta \in V$, where V is some neighborhood of the point z, the stalk $\mathfrak{F}_\zeta$ is generated by a finite system of sections $\{f^i\}$ of the sheaf $\mathfrak{F}(Z)$ over V and the stalk $\mathfrak{G}_\zeta$ by a finite system of sections $\{g^i\}$ of the sheaf $\mathfrak{G}(Z)$ over V.

Because of the permissibility of the homomorphism ϕ_z we have

$$\varphi_z(f^i_z) = \sum_j \lambda^i_j g^j_z, \qquad i = 1, 2, \ldots . \tag{2.15}$$

Here the coefficients λ^i_j are holomorphic functions in some neighborhood U of the point z.

We further assume that the neighborhood $U \subset V$ and that therein the germs g^j_z in equation (2.15) may be replaced by the holomorphic functions representing them.

From the coherency of the sheaf $\mathfrak{F}(Z)$ it further follows that the module of relations $R_\zeta(f^1, f^2, \cdots)$ at every point $\zeta \in V$ is generated by a finite collection of systems $\{a^1_k, a^2_k, \cdots\}$, where all $a^i_k \in \mathfrak{O}_V$. At the point $\zeta \in V$ we have

$$\sum_i a^i_k f^i = 0. \tag{2.16}$$

From equations (2.15) and (2.16) it follows that at the point $\zeta \in U$

$$\sum_{i,\,j} a^i_k \lambda^i_j g^j = 0. \tag{2.17}$$

We consider the mappings defined at $\zeta \in U$ by the equations

$$\varphi_\zeta(f^i_\zeta) = \sum_j \lambda^i_j g^j_\zeta, \qquad i = 1, 2, \ldots . \tag{2.15$_1$}$$

We show that formula (2.15$_1$) defines the desired analytic homomorphism if we establish that $\Sigma_{i,j} b_i \lambda^i_j g^j_\zeta = 0$ whenever $\Sigma_i b_i f^i_\zeta = 0$ (where b_i are certain functions holomorphic at the point ζ).

But in our case $b_i = \Sigma_k \beta_k a^i_k$, where β_k are functions holomorphic at the point ζ. Therefore we have

$$\sum_{i,\,j} b_i \lambda^i_j g^j_\zeta = \sum_k \Big(\sum_{i,\,j} a^i_k \lambda^i_j g^j_\zeta \Big) \beta_k = 0.$$

LEMMA 3. *Let* $\mathfrak{F}(Z)$ *and* $\mathfrak{G}(Z)$ *be coherent analytic sheaves over a complex manifold* Z. *Then for any analytic homomorphism* $\phi: \mathfrak{F}(K) \to \mathfrak{G}(K)$, *where* K *is a certain compactum, one can find a neighborhood* V *of the compactum* K *and an analytic homomorphism* $\tilde{\phi}: \mathfrak{F}(V) \to \mathfrak{G}(V)$ *which is an extension of the homomorphism* ϕ.

PROOF. By Lemma 2, in some neighborhood W of an arbitrary point $z \in K$ there exists an analytic homomorphism $\psi: \mathfrak{F}(W) \to \mathfrak{G}(W)$ satisfying the condition $\psi_z = \phi_z$. By Lemma 1 one can choose this neighborhood W in such a way that the equation $\psi_\zeta = \phi_\zeta$ holds at every point $\zeta \in W \cap K$. The compactum K can be covered by a finite system of similar neighborhoods W_i of points $z_i \in K$. In each of these neighborhoods there will be defined an analytic homomorphism ψ^i: $\mathfrak{F}(W_i) \to \mathfrak{G}(W_i)$ satisfying the condition $\psi^i_\zeta = \phi_\zeta$ at all points $\zeta \in W_i \cap K$. We consider a system of domains $\{V_i\}$, where $\overline{V}_i \subset W_i$, covering the whole of the compactum K. The set $A = \bigcup_i \overline{V}_i \supset K$. Suppose that for $i \in I_z$ the point $z \in A$ belongs to $\overline{V}_i$. The set of those points $z \in A$ for which the homomorphisms ψ^i_z coincide among themselves for all $i \in I_z$ is open in A and contains the compactum K. We can single out from this set the desired neighborhood V of the compactum K.

COMPLETION OF THE PROOF OF THEOREM 9.1. By the condition of coherency of the sheaf $\mathfrak{F}(K)$ we can place in correspondence with each point $z \in K$ its neighborhood U and an analytic coherent sheaf $\mathfrak{G}(U)$ such that the sheaves $\mathfrak{F}(K \cap U)$ and $\mathfrak{G}(K \cap U)$ turn out to be analytically isomorphic. The compactum K may be covered by a finite system of similar neighborhoods U_i of points $z_i \in K$. In each of these neighborhoods there is defined an analytic coherent sheaf $\mathfrak{G}^i(U_i)$ and an analytic isomorphism $\psi^i: \mathfrak{F}(K \cap U_i) \to \mathfrak{G}^i(K \cap U_i)$. We consider a system of domains $\{V_i\}$, where $\overline{V}_i \subset W_i$, which together cover the whole of the compactum K. Let $A = \bigcup_i \overline{V}_i \subset K$.

From the manner of defining the sheaf $\mathfrak{G}^i(U_i)$ it follows that: if the intersection $K \cap \overline{V}_i \cap \overline{V}_j = K_{ij} \neq \emptyset$, then there exists an analytic isomorphism $\psi^{ij} = (\psi^i)^{-1} \circ \psi^j$: $\mathfrak{G}^j(K_{ij}) \to \mathfrak{G}^i(K_{ij})$; if here the intersection $K \cap \overline{V}_i \cap \overline{V}_j \cap \overline{V}_k = K_{ijk} \neq \emptyset$, then $\psi^{ij} \circ \psi^{jk} \circ \psi^{ki} = 1$.

By Lemma 3, in some neighborhood $U_{ij} \subset U_i \cap U_j$ of the intersection K_{ij} there exists a homomorphism ϕ^{ij}: $\mathfrak{G}^j(U_{ij}) \to \mathfrak{G}^i(U_{ij})$ which is an extension of the homomorphism ψ^{ij}. We set $U_{ii} = U_i$. Note that by definition $\psi^{ii} = 1$; we also set $\phi^{ii} = 1$.

Every point $z \in A$ that is sufficiently close to the compactum K has the following properties: when $z \in \overline{V}_i \cap \overline{V}_j$, then $z \in U_{ij}$ (and thus the homomorphism ϕ^{ij} is defined at that point z); if $z \in \overline{V}_i \cap \overline{V}_j \cap \overline{V}_k$, then $(\phi^{ij} \circ \phi^{jk} \circ \phi^{ki})_z = 1$.

Let $V \subset A$ be a set of points with these properties. On each intersection $V \cap \overline{V}_i$ there is defined the sheaf $\mathfrak{G}^i(V \cap \overline{V}_i)$. On each nonempty intersection of the sets $V \cap \overline{V}_i$ and $V \cap \overline{V}_j$ there is defined the analytic isomorphism ϕ^{ij} of the sheaves $\mathfrak{G}^i$ and $\mathfrak{G}^j$, satisfying the condition of transitivity $\phi^{ij} \circ \phi^{jk} \circ \phi^{ki} = 1$. These sheaves together define a coherent analytic sheaf $\mathfrak{G}(V)$. In addition the sheaves $\mathfrak{G}(K)$ and $\mathfrak{F}(K)$ are analytically isomorphic. Thus Theorem 9.1 is proved.

2. Complex manifolds having properties (A) and (B).

DEFINITION (*property* (A)). A complex manifold Z or a compactum Z *has the property* (A) if for any coherent analytic sheaf $\mathfrak{F}(Z)$ and at each point $z \in Z$ the image of the group $H^0(Z, \mathfrak{F})$ in the group $\mathfrak{F}_z$ generates the whole group $\mathfrak{F}_z$ as an $\mathfrak{O}_z$-module.

Recall that by Theorem 0.2 the zero-dimensional cohomology group $H^0(Z, \mathfrak{F})$ with coefficients in the sheaf $\mathfrak{F}(Z)$ coincides with the group $\mathfrak{F}_Z$ of sections of the sheaf $\mathfrak{F}(Z)$ over the manifold Z.

DEFINITION (*property* (B)). A complex manifold Z or a compactum Z in some complex manifold *has the property* (B) if for any coherent analytic sheaf $\mathfrak{F}(Z)$ the cohomology group $H^q(Z, \mathfrak{F}) = 0$ for all $q \geq 1$.

From Theorems 8.13 and 8.14 follows the

THEOREM 9.2. *If a complex manifold Z (or a compactum $K \subset Z$) has the properties* (A) *and* (B), *then so does any properly imbedded complex manifold $E \subset Z$ (or intersection $K \cap E$).*

Later the following theorem will be used frequently.

THEOREM 9.3. *If a compactum $K \subset Z$ has in the complex manifold Z a fundamental system of compact neighborhoods $\{K_n\}$, each of which has the properties* (A) *and* (B), *then the compactum K itself also has the properties* (A) *and* (B).

PROOF. By Theorem 9.1 any coherent analytic sheaf $\mathfrak{F}(K)$ can be extended to some neighborhood $V \subset K$. We may assume that $K_n \subset V$ for every number n. Let K' be one of the compacta K_n. By assumption, the image of the group $H^0(K', \mathfrak{F})$ in the group $\mathfrak{F}_z$ generates the whole of this group as an $\mathfrak{O}_z$-module at any point $z \in K'$. Hence it follows that the image of the group $H^0(K, \mathfrak{F})$ at

points $z \in K$ has the same property, i.e., the compactum K has the property (A). Our assertion concerning the property (B) follows from the equation: $H^q(K, \mathfrak{F}) = \lim \operatorname{ind} H^q(K_n, \mathfrak{F})$.

THEOREM 9.4. *If $\mathfrak{F}(Z)$ is a coherent analytic sheaf over a complex manifold Z (or a compactum Z in a similar manifold) with the property* (A), *and if $f^i \in H^0(Z, \mathfrak{F})$, $i = 1, \cdots, q$, is a finite system of sections of the sheaf $\mathfrak{F}(Z)$ over Z, then the $\mathfrak{D}_z$-module of relations among the functions f^i at any point $z \in Z$ is generated by the $\mathfrak{D}_Z$-module of relations among the functions f^i over the entire Z.*

In other words, if the relation $\Sigma_i a_i f^i = 0$ holds in a neighborhood of a point $z \in Z$, where a_i, $i = 1, \cdots, q$, are functions holomorphic in that neighborhood of the point z, then there exist a finite system of functions $b_i^j \in \mathfrak{D}_Z$, $i = 1, \cdots, q$, $j = 1, \cdots, p$ and a finite system of functions $c_j \in \mathfrak{D}_z$, $j = 1, \cdots, p$ such that $a_i = \Sigma_j b_i^j c_j$, $i = 1, \cdots, q$, in the neighborhood of the point z and $\Sigma_i b_i^j f^i = 0$ for $j = 1, \cdots, p$ at every point of the manifold (or compactum) Z.

PROOF OF THEOREM 9.4. The sheaf of relations $R_Z(f^1, \cdots, f^q)$ is coherent in view of Theorem 8.10. Therefore, because of the property (A), this sheaf has the property stated in Theorem 9.4.

THEOREM 9.5. *If $\mathfrak{G}(Z)$ is a sheaf over a manifold Z (or a compactum Z in a similar manifold) and $\mathfrak{F}(Z)$ is a coherent analytic sheaf over Z with the property* (B), *while the sheaf $\mathfrak{F}(Z)$ is a subsheaf of the sheaf $\mathfrak{G}(Z)$, then the canonical homomorphism*

$$\varphi:\ H^0(Z, \mathfrak{G}) \rightarrow H^0(Z, \mathfrak{G}/\mathfrak{F})$$

surjectively maps the first module onto the second.

In other words, for every section h of the factor sheaf $\mathfrak{G}(Z)/\mathfrak{F}(Z)$ over Z one can find a section g of the sheaf $\mathfrak{G}(Z)$ over Z such that the section h corresponds to the section g under the homomorphism ϕ.

PROOF OF THEOREM 9.5. In view of Theorem 0.3 the sequence

$$H^0(Z, \mathfrak{G}) \xrightarrow{\varphi} H^0(Z, \mathfrak{G}/\mathfrak{F}) \rightarrow H^1(Z, \mathfrak{F}) \tag{2.18}$$

is exact. Because of the property (B) the group $H^1(Z, \mathfrak{F}) = 0$. Therefore the second homomorphism carries any element of the group $H^0(Z, \mathfrak{G}/\mathfrak{F})$ into zero. From the definition of an exact sequence it follows that an elementary group $H^0(Z, \mathfrak{G}/\mathfrak{F})$ is an image of some element of the group $H^0(Z, \mathfrak{G})$. From this our assertion follows.

THEOREM 9.6. *Let $\mathfrak{F}(Z)$ be a coherent analytic sheaf over a complex manifold Z (or a compactum Z in a similar manifold) having the property* (B). *If a finite system of elements $f^i \in H^0(Z, \mathfrak{F})$, $i = 1, \cdots, p$, generates an $\mathfrak{O}_z$-module $\mathfrak{F}_z$ at each point $z \in Z$, then this system generates the group $H^0(Z, \mathfrak{F})$ as an $\mathfrak{O}_Z$-module.*

PROOF. Let ϕ_z: $\mathfrak{O}_z^p \longrightarrow \mathfrak{F}_z$ be a homomorphism associating with an element $(c_1, \cdots, c_p) \in \mathfrak{O}_z^p$ an element $\Sigma_i c_i f^i \in \mathfrak{F}_z$. By assumption, the image of this homomorphism is the entire stalk $\mathfrak{F}_z$. The collection of homomorphisms ϕ_z at all points $z \in Z$ defines a homomorphism ϕ: $\mathfrak{O}^p(Z) \longrightarrow \mathfrak{F}(Z)$. Let $\mathring{R}(Z) = \operatorname{Ker} \phi$. Then the sheaf $\mathfrak{F}(Z)$ is isomorphic to the factor sheaf $\mathfrak{O}^p(Z)/R(Z)$ and the sheaf $R(Z)$ is coherent by Theorem 8.9. On the other hand, the homomorphism ϕ defines a homomorphism of modules of sections

$$\Phi : \quad H^0(Z, \mathfrak{O}^p(Z)) \to H^0(Z, \mathfrak{F}),$$

associating with an element $(c_1, \cdots, c_p) \in \mathfrak{O}_Z^p$ an element $\Sigma_i c_i f^i \in \mathfrak{F}_Z$. In view of Theorem 9.5 the homomorphism Φ has the entire group $H^0(Z, \mathfrak{F})$ as its image. Theorem 9.6 is proved.

EXAMPLE. Let $\mathfrak{F}(Z) = \mathfrak{O}(Z)$. Then a finite system $(f^1, \cdots, f^p) \in \mathfrak{O}_Z^p$ satisfies the assumption of Theorem 9.6, if the functions $f^1, \cdots, f^p$ have no common zero on Z. In this case, by Theorem 9.6, they generate the module of all functions holomorphic on the whole manifold (or compactum) Z. In particular, for a proper choice of coefficients $c_1, \cdots, c_p \in \mathfrak{O}_Z$, the identity

$$\sum_i c_i f^i \equiv 1 \quad \text{for} \quad z \in Z$$

holds.

Moreover, it is to be noted that we easily obtain from Theorem 9.6 the

COROLLARY 1. *Let Z be a compactum with the property* (B). *If an $\mathfrak{O}_Z$-submodule M of the module $H^0(Z, \mathfrak{F})$: 1) generates the stalk $\mathfrak{F}_z$ at every point $z \in Z$, 2) contains a submodule M' with a finite set of generators which also generates the stalk $\mathfrak{F}_z$ at every point $z \in Z$, then $M = H^0(Z, \mathfrak{F})$.*

In fact, in view of Theorem 9.6, $M' = H^0(Z, \mathfrak{F})$ and consequently $M = H^0(Z, \mathfrak{F})$.

From the same theorem there follows the

COROLLARY 2. *If Z is a compactum with the properties* (A) *and* (B), *then the group $H^0(Z, \mathfrak{F})$ is a module with a finite set of generators.*

In fact, since the compactum Z has the property (A), one can find at each

point $z \in Z$ a neighborhood $U_z \subset Z$ and a finite system of elements of the group $H^0(Z, \mathfrak{F})$ generating the stalk $\mathfrak{F}_\zeta$ at any point $\zeta \in U_z$. We take a finite set of similar neighborhoods U_z covering the compactum Z; as a result we form a finite system of elements of the group $H^0(Z, \mathfrak{F})$ generating the stalk $\mathfrak{F}_\zeta$ in the structure of the $\mathfrak{O}_\zeta$-module at each point $\zeta \in Z$. In view of Theorem 9.6 this system of elements will generate the group $H^0(Z, \mathfrak{F})$ in the structure of the $\mathfrak{O}_Z$-module.

THEOREM 9.7. *Let E be a closed compact submanifold of a complex manifold Z and K be a compactum in Z. If the compactum K has the property* (B), *then all E-holomorphic functions on the intersection $E \cap K$ are induced by Z-holomorphic functions on K.*

PROOF. Indeed, let $\mathfrak{S}(Z)$ be the sheaf of the submanifold E (see §8.6), and let ${}_Z\mathfrak{O}(K)$ be the sheaf of germs of functions Z-holomorphic on the compactum K. In view of Theorem 8.14 the sheaf $\mathfrak{S}(Z)$ is coherent. The sheaf ${}_E\mathfrak{O}(Z)$ coincides with the factor sheaf ${}_Z\mathfrak{O}(K)/\mathfrak{S}(K)$. By Theorem 9.5 every section of the factor sheaf ${}_Z\mathfrak{O}(K)/\mathfrak{S}(K)$ over the compactum K is an image of some section of the sheaf ${}_Z\mathfrak{O}(K)$ over that compactum. But this is just the assertion of Theorem 9.7.

3. Formulation of H. Cartan's theorems on holomorphically complete complex manifolds. In §18, Chapter III, (I) we considered holomorphically complete complex manifolds (or Stein manifolds). We have shown there that a complex manifold with a countable basis of neighborhoods having the following properties is holomorphically complete:

(a) holomorphic separability (i.e., for each two distinct points $z', z'' \in Z$ there exists a function $f(z) \in \mathfrak{O}_Z$ such that $f(z') \neq f(z'')$);

(b) at each point $z \in Z$ there exists a system of functions holomorphic on the manifold Z and playing, in some neighborhood of the point z, the role of local coördinates;

(c) holomorphic convexity (i.e., the holomorphically convex hull K of each compactum $K \subset Z$ is a compactum).

Further we shall use the fact that in view of what has been said the holomorphically complete complex manifold is always the union of a countable set of compacta.

Already we have noted above that in older papers all of these properties were included in the definition of holomorphically complete complex manifolds. However, it has been shown later (by H. Grauert) that some of these properties follow

from the others.

Below we shall show (without using the results of H. Grauert; see Theorem 12.10) that when the properties (a) and (b) are present the condition (c) may be replaced by the equivalent condition:

(c′) for every sequence of points $z^\nu \in Z$ not having a limiting point on the manifold Z one can find a function $f(z) \in \mathfrak{O}_Z$ such that the sequence $|f(z^\nu)|$ will not be bounded.

We note that the condition (c′) evidently follows from the property (c). It is required to prove only the inverse assertion.

The following two theorems of H. Cartan occupy a central position in the theory of holomorphically complete complex manifolds:

THEOREM 9.8 (H. Cartan's Theorem (A)). *A holomorphically complete complex manifold Z (or a compactum Z in such a manifold which coincides with its hull, the hull being convex with respect to a family of functions, that are holomorphic on the whole of the manifold) possesses the property* (A).

In other words, *for any coherent analytic sheaf $\mathfrak{F}(Z)$ and at each point $z \in Z$ the image of the group $H^0(Z, \mathfrak{F})$ in $\mathfrak{F}_z$ generates the group $\mathfrak{F}_z$ as an $\mathfrak{O}_z$-module.*

THEOREM 9.9 (H. Cartan's Theorem (B)).[1] *The holomorphically complete complex manifold Z (or the compactum Z in a similar manifold which coincides with its hull, convex with respect to the family of functions, holomorphic on the whole of that manifold) possesses the property* (B).

In other words, *for any coherent analytic sheaf $\mathfrak{F}(Z)$ the group $H^q(Z, \mathfrak{F}) = 0$ for all $q \geq 1$.*

§§10 and 11 are devoted to the proof of Theorems (A) and (B). In §10 these are proved for cubes on the space C^n and in §11 the proof of the theorems is completed in the general case.

Theorems (A) and (B) can be extended to the holomorphically complete complex spaces (see Grauert [4]).

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1) As H. Cartan [4], (I) mentions, Theorem (B) in the cases $q > 1$ was first stated by J.-P. Serre. To him is also due the formulation of this theorem in the terminology of the cohomology group with coefficients in the sheaf. Concerning the proof of Theorems (A) and (B) see also: R. C. Gunning, *On Cartan theorems* A *and* B *in several complex variables*, Ann. Mat. Pura Appl. (4) 65 (1961), 1–11. An essential extension of Theorem (B) is constructed in A. Andreotti and H. Grauert, *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France 90 (1962), 193–259.

§10. PROOF OF H. CARTAN'S THEOREMS (A) AND (B) FOR CUBES ON THE SPACE C^n

1. Complex Lie groups.

DEFINITION. A *complex Lie group* is the name given to a (not necessarily connected) complex manifold $G = G^n$ on which there is defined a composition $g_1 \circ g_2 = g_3$ (where points $g_1, g_2, g_3 \in G$) if:

1) The set of points $g \in G$ forms a group with respect to the composition $\circ$.

2) The mapping $(g, g') \longrightarrow g \circ g'^{-1}$, where points $g, g' \in G$, is a homomorphic mapping of the manifold $G \times G$ onto the manifold G.

Here, as always, by g^{-1} we denote an element (point) of the manifold G inverse to the element (point) $g \in G$. Further we denote by e the unit (neutral) element of G.

We note that because of the second requirement of our definition a complex Lie group is always a topological group.

EXAMPLES. I. The complex projective group Π, defined on the space P^n, becomes a complex Lie group if we introduce the following topology: as the ϵ-neighborhood of a transformation $\pi_0 \in \Pi$ we take the collection of all elements $\pi \in \Pi$ for which

$$\max_{z \in P^n} \chi(\pi_0 z, \ \pi z) < \varepsilon$$

Here χ is the chordal distance between corresponding points of the space P^n (see §5, Chapter I, (I)).

II. Consider $L(p, C)$, the collection of invertible square matrices of order p (p rows and p columns) whose elements are complex numbers. Under the operation of multiplication of matrices the set $L(p, C)$ forms a group. In this set we introduce a topology: as the ϵ-neighborhood of the matrix $l^0 \in L(p, C)$ with elements $l^0_{ik} \in C$, $i, k = 1, \cdots, p$, we take the collection of matrices $l \in L(p, C)$ with elements $l_{ik} \in C$, $i, k = 1, \cdots, p$, for which

$$\left(\sum_{i,k=1}^{p} | l_{ik} - l^0_{ik} |^2 \right)^{\frac{1}{2}} < \varepsilon.$$

As a result we obtain the complex Lie group $GL(p, C)$.

In the theory of Lie groups it is established that in some neighborhood $U_e \subset G$ of the unit element $e \in G$ one can introduce the so-called *canonical*

coordinates $\zeta_1, \cdots, \zeta_n$.[1] The point e serves as the origin of these coordinates; if for elements $g = (\zeta_1, \cdots, \zeta_n) \in U_e$ and $g' = (a\zeta_1, \cdots, a\zeta_n) \in U_e$ the element $g + g' = (\zeta_1 + a\zeta_1, \cdots, \zeta_n + a\zeta_n)$ also belongs to U_e, then $g \circ g' = g + g'$. It can be shown that the presence of these two properties uniquely defines the canonical coordinate system up to the linear transformation

$$\zeta'_\nu = \sum_{\mu=1}^{n} \alpha_{\nu\mu}\zeta_\mu, \quad \nu = 1, \ldots, n.$$

Usually we take as the neighborhood U_e a ball $\{|g| < R\}$ of a certain radius R. Here $|g| = (|\zeta_1|^2 + \cdots + |\zeta_n|^2)^{1/2}$ is the *norm* of the element $g = (\zeta_1, \cdots, \zeta_n)$.

If the norm $|g|$ of an element $g \in U_e$ is sufficiently small, then its inverse element g^{-1} also belongs to U_e. If the norms of the elements $g^1 = (\zeta_1^1, \cdots, \zeta_n^1) \in U_e$ and $g^2 = (\zeta_1^2, \cdots, \zeta_n^2) \in U_e$ are sufficiently small, then the canonical coordinates of the elements $g^1 \circ g^2 \in U_e$ and $g^1 + g^2 = (\zeta_1^1 + \zeta_1^2, \cdots, \zeta_n^1 + \zeta_n^2) \in U_e$ can differ only by infinitesimal quantities of second order in the norms $|g^1|$ and $|g^2|$. By the use of this fact we can show that if the radius R of the neighborhood U_e is sufficiently small, and if elements $g^1, g^2, (g^1 + g^2), (g^1)^{-1}, (g^2)^{-1}$ as well as the element $(g^1)^{-1} \circ (g^1 + g^2) \circ (g^2)^{-1}$ belong to this neighborhood, then

$$|(g^1)^{-1} \circ (g^1 + g^2) \circ (g^2)^{-1}| \leq M \max(|g^1|^2, |g^2|^2). \tag{2.19}$$

Here M is some constant.

2. An auxiliary proposition.

DEFINITION (analytically contractible complex manifold). A complex manifold Z is said to be analytically contractible if there exists a continuous mapping $f(z, t)$: $Z \times I \longrightarrow Z$, where $z \in Z$, $t \in I = \{0 \leq t \leq 1\}$, having the following properties:

1) For each $t \in I$ the mapping $f_t(z)$: $Z \longrightarrow Z$ defined by the equation $f_t(z) = f(z, t)$, where $z \in Z$, is a holomorphic mapping of the manifold Z into itself.

2) $f(z, 0) = z$, $f(z, 1) = a$ for every point $z \in Z$. Here a is some fixed point of the manifold Z.

Now we can prove the following theorem.

THEOREM 10.1. *There are given :*

1) *simply-connected compact polycylindrical domains* $\overline{\mathfrak{D}}' = \overline{D}_1' \times \cdots \times \overline{D}_n' \subset C^n$ *and* $\overline{\mathfrak{D}}'' = \overline{D}_1'' \times \cdots \times \overline{D}_n'' \subset C^n$, *where* $D_i' = D_i''$ *for all indices* i, *except possibly one, and their intersection* $\overline{\mathfrak{D}}' \cap \overline{\mathfrak{D}}''$ *is simply-connected;*

1) See, for example, C. Chevalley, *Theory of Lie groups* I, Princeton Univ. Press, Princeton, N.J., 1946, Chapter IV, in particular §§1, 8, 10. See also Grauert [2].

2) *a complex connected Lie group* G. *Then there corresponds to each holomorphic mapping* $f\colon \overline{\mathfrak{D}}' \cap \overline{\mathfrak{D}}'' \longrightarrow G$, *holomorphic mappings* $f'\colon \overline{\mathfrak{D}}' \longrightarrow G$ *and* $f''\colon \overline{\mathfrak{D}}'' \longrightarrow G$, *such that* $f' \circ f'' = f$ *at points* $z \in \overline{\mathfrak{D}}' \cap \overline{\mathfrak{D}}''$. *Here the composition* $\circ$ *is to be taken by the rules fixed by the group* G.

EXAMPLE. Let $n = 1$, $G = C^*$, where C^* is the multiplicative group of complex numbers excluding zero. We set $f = e^{g(z)}$, where $g(z)$ is a holomorphic function in some neighborhood of the simply-connected compact domain $\overline{D}' \cap \overline{D}''$. By the use of Cauchy's integral formula, we may express the function $g(z)$ in the form of a sum

$$g(z) = g'(z) + g''(z) \text{ for } z \in \overline{\mathfrak{D}}'' \cap \overline{\mathfrak{D}}'',$$

where the functions $g'(z)$ and $g''(z)$ are holomorphic, respectively, in the compact domains $\overline{\mathfrak{D}}'$ and $\overline{\mathfrak{D}}''$. Now setting $f' = e^{g'(z)}$ and $f'' = e^{g''(z)}$, we obtain the assertion of our theorem.

REMARK. In the sequel, in applying Theorem 10.1 we always take $G = GL(p, C)$. In this case the roles of the holomorphic mappings f', f'' and f are played by the matrices M', M'' and M of order p, wnose elements are functions holomorphic in the closed domains $\overline{\mathfrak{D}}'$, $\overline{\mathfrak{D}}''$ and $\overline{\mathfrak{D}}' \cap \overline{\mathfrak{D}}''$, respectively. In view of Theorem 10.1, in our case $M'M'' = M$ for $z \in \overline{\mathfrak{D}}' \cap \overline{\mathfrak{D}}''$.

3. **Proof of the lemmas.**

LEMMA 1. *Let* E *be a (generally speaking, nonconnected) complex manifold which is a sum of analytically contractible complex manifolds* E_j, $j \in \mathfrak{J}$, *and let* F *be a linearly connected complex manifold. Then the space* $\mathcal{H}(E, F)$ *of holomorphic mappings* $E \longrightarrow F$, *in which there is introduced a compact-open topology,*[1] *is linearly connected.*

PROOF. It is readily seen that the space $\mathcal{H}(E, F)$ is homeomorphic to the product of the spaces $\mathcal{H}(E_j, F)$, $j \in \mathfrak{J}$. Therefore it suffices to prove the linear connectedness of each space $\mathcal{H}(E_j, F)$; in other words, without loss of generality, we may assume in our proof that the manifold E itself is analytically contractible. Let $f(z, t)$ be a mapping realizing the analytic contraction of the manifold E.

1) Let X and Y be topological spaces, let Map (X, Y) be the set of all continuous mappings $X \to Y$, let a set $K \subset X$ be compact, and let a set $U \subset Y$ be open. We denote by $S(K, U) \subset$ Map (X, Y) a set of mappings $f\colon X \to Y$ for which $f(K) \subset U$. We introduce a topology in the set Map (X, Y) by taking as the basis of open sets the collection of finite intersections of sets $S(K, U)$. Such a topology is said to be *compact-open*. This topology may be also introduced by starting from the concept of a uniform convergence of mappings (defined by means of compact sets; see §3.4, Chapter I).

Then $h(f(z, t))$, where $h \in \mathcal{H}(E, F)$, is a continuous mapping $E \times I \longrightarrow F$, holomorphic (in z) for every fixed value of the parameter t. It carries each point $z \in E$ into the point $h(z) \in F$ for $t = 0$ and all points $z \in E$ into the point $h(a) \in F$ for $t = 1$. Thus there is defined for every value of the parameter $t \in I$ a mapping h_t: $E \longrightarrow F$, i.e., there is defined a point h_t of the space $\mathcal{H}(E, F)$. We shall show that these points form a path in the space $\mathcal{H}(E, F)$. In other words, we have to establish that for every number $t_0 \in I$ and a neighborhood $V \subset \mathcal{H}(E, F)$ of the element $h_{t_0} \in \mathcal{H}(E, F)$ there exists a number $\epsilon > 0$ such that the point $h_t \in V$ for $|t - t_0| < \epsilon$ and $t \in I$. The neighborhood V may be assumed to belong to the basis of the compact-open topology, i.e., we may put $V = S(K, U)$, where the set $K \subset E$ is compact and the set $U \subset F$ is open. Under a continuous mapping the inverse image of the open set is open; therefore the inverse image of the set U in the space $E \times I$ under the mapping $E \times I \longrightarrow F$ is an open set. The intersection of this inverse image with the product $K \times I$ is a neighborhood of $K \times t_0$ in $K \times I$. Since K is compact, this neighborhood contains a set of the form $K \times J$, where $J \subset I$ is a certain interval containing the point t_0. Hence it follows that $h_t \in V$ for $t \in J$.

This means that in the space $\mathcal{H}(E, F)$ there is a path connecting an arbitrary point h with a constant mapping. Thus the space $\mathcal{H}(E, F)$ is seen to be connected, since any two constant mappings $E \longrightarrow F$ can be joined to each other in the space $\mathcal{H}(E, F)$. This last result follows from the linear connectedness of the space F. The lemma is proved.

Let G be a complex Lie group, and let $\mathcal{H}(E, G)$ be the space of holomorphic mappings of the complex manifold E into the group G. Then the law of composition in the group G induces the law of composition in the space $\mathcal{H}(E, G)$. As a result this space becomes a topological group. The unit element of this group is the mapping which transforms the whole manifold E into the unit element e of the group G.

If E is a sum of analytically contractible complex manifolds and if the group G is linearly connected, then the group $\mathcal{H}(E, G)$ is linearly connected, by Lemma 1. Therefore, *if* $V \subset \mathcal{H}(E, G)$ *is some neighborhood of the unit element* ϵ, *every element* $h \in \mathcal{H}(E, G)$ *may be represented as the composition of a finite set of elements belonging to the neighborhood* V.

Indeed, let $V_g = g \circ V$, where g is an arbitrary element of the group G and $\circ$ is the law of composition of the group G. The collection $\{V_g, g \in G\}$ covers the group G. Consider an arbitrary element $g_0 \in G$. In view of the linear con-

nectedness of the group G one can join it to the element ϵ by a certain path $\mathfrak{L}$. We choose a finite collection of elements $g_i \in G$, $i = 1, \cdots, k$; $V_{g_i} \cap V_{g_{i+1}} \neq \emptyset$, $i = 1, \cdots, k-1$, in such a way that $g_1 = \epsilon$, $g_k = g_0$, $\bigcup_{(i)} V_{g_i} \supset \mathfrak{L}$. Take any element $g \in V_{g_i} \cap V_{g_{i+1}}$; it can be represented in the form $g = g_i \circ \gamma_i = g_{i+1} \circ \mathfrak{g}_{i+1}$, where γ_i, $\mathfrak{g}_{i+1} \in V$. Hence it is also found that $g_{i+1} = g_i \circ \gamma_i \circ \mathfrak{g}_{i+1}^{-1}$. By iteration we obtain:

$$g_0 = g_k = g_{k-1} \circ \gamma_{k-1} \circ \mathfrak{g}_k^{-1} = g_1 \circ \mathfrak{g}_k^{-1} \circ \gamma_2 \circ \mathfrak{g}_3^{-1} \circ \cdots \circ \gamma_{k-1} \circ \mathfrak{g}_k^{-1}.$$

LEMMA 2. *Let $K \subset C^n$ be a compact simply-connected polycylindrical domain. Every holomorphic mapping $f\colon K \longrightarrow G$, where G is a complex Lie group, can be uniformly approximated by holomorphic mappings $C^n \longrightarrow G$.*

PROOF. Let $U \subset G$ be a sufficiently small neighborhood of the unit element of the group G such that there exists in it a local system of complex coordinates. Evidently any compact simply-connected polycylindrical domain is analytically contractible; therefore the mapping $f\colon K \longrightarrow G$ can be represented as the composition of a finite set of mappings $f_i\colon K \longrightarrow U$, $i = 1, 2, \cdots$. These mappings belong to the neighborhood of the identity mapping in the space $\mathcal{H}(E, G)$ (in the compact-open topology defined above). It suffices to prove that each of the mappings f_i can be uniformly approximated by holomorphic mappings $C^n \longrightarrow U$ (for the definition of such an approximation see §3.5, Chapter I). The local coordinates of the point $f_i(z) \in U$ are holomorphic functions of the point z in the domain K. The assertion of the lemma follows from the fact that such functions may be uniformly approximated by polynomials (in this connection see Example 1 in §3.2, Chapter I).

4. **Completion of the proof of Theorem 10.1.** Let us introduce some notations. We assume that $D_i' = D_i'' = D_i$ for $i = 2, \cdots, n$. We further set $z_1 = z$, $D_1' = D'$, $D_1'' = D''$, $D = D' \cap D''$, $\mathfrak{D} = D_2 \times \cdots \times D_n$. Then $\mathfrak{D}' = D' \times \mathfrak{D}$ and $\mathfrak{D}'' = D'' \times \mathfrak{D}$. By the condition of Theorem 10.1 the domains $D, D', D'', D_2', \cdots, D_n'$ are simply-connected and $\overline{\mathfrak{D}}$ is a compact simply-connected polycylinder in the space $C_{\mathfrak{z}}^{n-1}$. Here $\mathfrak{z} = (z_2, \cdots, z_n)$.

Let $D_0' \supset \overline{D}'$, $D_0'' \supset \overline{D}''$ and $\mathfrak{D}_0 \supset \overline{\mathfrak{D}}$ be respectively neighborhoods of the closed domains $\overline{D}'$, $\overline{D}''$ and $\overline{\mathfrak{D}}$ such that the mapping $f\colon \overline{\mathfrak{D}}' \cap \overline{\mathfrak{D}}'' \longrightarrow G$ can be holomorphically extended to the domain $\overline{D}_0 \times \overline{\mathfrak{D}}_0$, where $D_0 = D_0' \cap D_0''$. We may assume that:

1) $\overline{\mathfrak{D}}_0$ is a compact simply-connected polycylinder;

2) D_0' and D_0'' are unions of squares with sides parallel to the coordinate

axes; the coordinates of vertices of these squares are integral multiples of some number r;

3) the domain D_0 is simply-connected.

The boundaries of the domains D_0', D_0'', D_0 are unions of segments parallel to the coordinate axes. We denote by $D_0'(a)$, $D_0''(a)$, $D_0(a)$ the domains obtained from the domains D_0', D_0'', D_0 by cutting off a strip of width equal to a along all boundaries. Then $D_0(a) = D_0'(a) \cap D_0''(a)$. The number $a < r/2$ is chosen so small that $\overline{D}' \subset D_0'(a)$ and $\overline{D}'' \subset D_0''(a)$. We set $D'_{(i)} = \overline{D_0'(a - 2^{-i}a)}$, $D''_{(i)} = \overline{D_0''(a - 2^{-i}a)}$, $D_{(i)} = D_0(a - 2^{-i}a)$, $\gamma_i' = D''_{(i)} \cap \partial D'_{(i)}$ and $\gamma_i'' = D'_{(i)} \cap \partial D''_{(i)}$. Here $i \geq 1$ is an integer. Evidently

$$\partial(D_{(i)}) = \partial\left(D'_{(i)} \cap D''_{(i)}\right) = \gamma_i' \cup \gamma_i'',$$

the distances between sets γ_i' and $D'_{(i+1)}$, γ_i'' and $D''_{(i+1)}$ are not less than $2^{-(i+1)}a$, and the length of the broken lines γ_i' and γ_i'' are bounded from above by a number independent of the index i. We denote by κ the quotient of this number divided by π.

Let $U \subset G$ be a neighborhood of the unit element e of the group G, in which a canonical coordinate system $t^1, \cdots, t^N$ is introduced and the relation (2.19) holds for $|g_1|$, $|g_2| < \alpha$. We may assume that

$$\alpha \leq \frac{a}{4M\kappa}.$$

We have the

LEMMA 3. *If a holomorphic mapping $f\colon \overline{D}_0 \times \overline{\mathfrak{D}}_0 \longrightarrow G$ transforms the entire domain $\overline{D}_0 \times \overline{\mathfrak{D}}_0$ into the neighborhood U and*

$$|f(z, \mathfrak{z})| \leqslant \frac{a\alpha}{\kappa} \quad \textit{for} \quad z \in \overline{D}_0,\ \mathfrak{z} \in \overline{\mathfrak{D}}_0, \tag{2.20}$$

then there exist holomorphic mappings $f'\colon \overline{\mathfrak{D}}' \longrightarrow U$ and $f''\colon \overline{\mathfrak{D}}'' \longrightarrow U$ such that $f = f' \circ f''$ for $(z, \mathfrak{z}) \in \overline{\mathfrak{D}}' \cap \overline{\mathfrak{D}}''$.

PROOF. Set $a\alpha/\kappa = \rho$ for brevity. Denote by $f_0^j(z, \mathfrak{z})$, $j = 1, \cdots, N$, the canonical coordinates of the mapping $f(z, \mathfrak{z}) = f_0(z, \mathfrak{z})$. We have $|f_0(z, \mathfrak{z})| \leq \rho$.

Further we define by induction on $i \geq 1$ some holomorphic functions $\{f_i^j(z, \mathfrak{z}), j = 1, \cdots, N\}$ in each domain $\overline{D}_{(i)} \times \overline{\mathfrak{D}}_0'$. Suppose that the functions $\{f_{i-1}^j(z, \mathfrak{z}), j = 1, \cdots, N\} = f_{i-1}(z, \mathfrak{z})$ have already been defined and that they satisfy the inequality $|f_{i-1}(z, \mathfrak{z})| \leq \kappa^{-i+1}\rho$ for $(z, \mathfrak{z}) \in \overline{D}_{(i-1)} \times \overline{\mathfrak{D}}_0$. Then, for

the indicated values of z and $\mathfrak{z}$, we have

$$f^j_{i-1}(z, \mathfrak{z}) = f'^j_{i-1}(z, \mathfrak{z}) + f''^j_{i-1}(z, \mathfrak{z}), \tag{2.21}$$

where

$$\left.\begin{aligned} f'^j_{i-1}(z, \mathfrak{z}) &= \frac{1}{2\pi i} \int\limits_{\gamma'_{i-1}} \frac{f^j_{i-1}(\zeta, \mathfrak{z})\, d\zeta}{\zeta - z}, \\ f''^j_{i-1}(z, \mathfrak{z}) &= \frac{1}{2\pi i} \int\limits_{\gamma''_{i-1}} \frac{f^j_{i-1}(\zeta, \mathfrak{z})\, d\zeta}{\zeta - z}, \end{aligned}\right\} \tag{2.22}$$

$j = 1, \cdots, N$. Here the functions $f'^j_{i-1}(z, \mathfrak{z})$ are defined and are holomorphic for $z \in \overline{D}'_{(i)}$, $\mathfrak{z} \in \overline{\mathfrak{D}}_0$ and satisfy the inequality

$$|f'_{i-1}(z, \mathfrak{z})| \leqslant \frac{\varkappa}{2}\left(4^{-(i-1)}\rho\right)\frac{2^i}{a} \leqslant 2^{-(i-1)}\alpha \tag{2.23}$$

there; the functions $f''^j_{i-1}(z, \mathfrak{z})$ are defined and are holomorphic for $z \in \overline{D}''_{(i)}$, $\mathfrak{z} \in \overline{\mathfrak{D}}_0$ and satisfy the inequality

$$|f''_{i-1}(z, \mathfrak{z})| \leqslant 2^{-(i-1)}\alpha \tag{2.24}$$

there. We set

$$f_i(z, \mathfrak{z}) = \left(f'_{i-1}(z, \mathfrak{z})\right)^{-1} \circ f_{i-1}(z, \mathfrak{z}) \circ \left(f''_{i-1}(z, \mathfrak{z})\right)^{-1} \tag{2.25}$$

for $z \in \overline{D}_{(i)}$ and $\mathfrak{z} \in \overline{\mathfrak{D}}_0$. Here $f'_{i-1} = \{f'^1_{i-1}, \cdots, f'^N_{i-1}\}$, $f''_{i-1} = \{f''^1_{i-1}, \cdots, f''^N_{i-1}\}$. For the indicated values of z and $\mathfrak{z}$ the canonical coordinates $f^j_i(z, \mathfrak{z})$, $j = 1, \cdots, N$, of the mapping $f_i(z, \mathfrak{z}) \in U$ are holomorphic functions which, in view of relations (2.21) and (2.19), satisfy the condition $|f_i(z, \mathfrak{z})| \leq M(2^{-i+1}\alpha)^2 \leq 4^{-i}\rho$.

Thus, since the mapping $f_0(z, \mathfrak{z})$ satisfies the condition (2.20), formulas (2.25) for $i = 1, 2, \cdots$ define the functions $f^j_i(z, \mathfrak{z})$ for all indices i; here formulas (2.21) define the holomorphic functions $f'^j_i(z, \mathfrak{z})$ and $f''^j_i(z, \mathfrak{z})$ respectively in the domains $\overline{D}'_{(i+1)} \times \overline{\mathfrak{D}}_0$ and $\overline{D}''_{(i+1)} \times \overline{\mathfrak{D}}_0$. In view of the inequality (2.23) the composition $f'_0 \circ \cdots \circ f'_i$ for $i \to \infty$ converges in the intersection $\bigcap_i [D'_{(i)} \times \mathfrak{D}_0]$, which is a neighborhood of the domain $\overline{D}' \times \overline{\mathfrak{D}} = \overline{\mathfrak{D}}'$, to a holomorphic mapping $f' : \overline{\mathfrak{D}}' \to U$ (it must be noted that the origin of the coordinates of the canonical coordinate system in the neighborhood U corresponds to the unit element of the group G). Analogously, on the basis of the inequality (2.24), we conclude that the composition $f''_i \circ \cdots \circ f''_0$ for $i \to \infty$ converges, in some neighborhood of the domain $\overline{D}'' \times \overline{\mathfrak{D}} = \overline{\mathfrak{D}}''$, to a holomorphic mapping $f'' : \overline{\mathfrak{D}}'' \to U$.

Consider the identity

$$f = (f_0' \circ \cdots \circ f_j') \circ f_{j+1} \circ f_j'' \circ \cdots \circ f_0'').$$

This follows from relations (2.25) for $i = 2, \cdots, j$ and holds in the intersection $\bigcap_i \overline{D}_{(i)} \times \overline{\mathfrak{D}}_0$. Taking the limit as $j \to \infty$ and observing that the point $f_{j+1}(z, \mathfrak{z})$ tends uniformly to the origin of the canonical coordinate system (which corresponds to the unit element of the group), we establish that $f = f' \circ f''$ for $(z, \mathfrak{z}) \in \overline{\mathfrak{D}}' \times \overline{\mathfrak{D}}''$.

Thus the proof of Lemma 3 is finished. Now we pass to the completion of the proof of Theorem 10.1. In view of Lemma 2 the mapping f considered in this theorem can be represented in the form of a composition $f_1 \circ g$, where the holomorphic mapping $f_1 \colon D_0 \times \mathfrak{D}_0 \to U$ satisfies the condition (2.20), while the mapping g is the restriction of some holomorphic mapping $C^n \to G$. In view of Lemma 3 we have: $f_1 = f_1' \circ f_1''$ for $(z, \mathfrak{z}) \in \overline{\mathfrak{D}}' \times \overline{\mathfrak{D}}''$, where $f_1' \colon \overline{\mathfrak{D}}' \to U$ and $f_1'' \colon \overline{\mathfrak{D}}'' \to U$. Setting $f' = f_1'$, $f'' = f_1'' \circ g$ we find that $f = f' \circ f''$ for $(z, \mathfrak{z}) = (z_1, \cdots, z_n) \in \overline{\mathfrak{D}}' \times \overline{\mathfrak{D}}''$. Thus the proof of Theorem 10.1 is completed.

5. **Proof of Theorem (A) for cubes in the space C^n.** On r coordinate axes of the space $R_{2n} = C^n$ we take equal closed intervals, not reducing to one point. Their product is a compact, topologically r-dimensional cube X. The embracing space C^n induces on it a complex structure and defines the sheaf $\mathfrak{O}(X)$ of germs of holomorphic functions.

We shall prove Theorems (A) and (B) for the cube X by induction with respect to its topological dimension; in this induction process we use Theorems (A) and (B) for the dimension $r-1$ to prove Theorems (A) and (B) for the dimension r. Theorem (A) for the cube X will be proved in the following formulation:

THEOREM 10.2 (**H. Cartan's Theorem (A′) for the cube X**). *For any coherent analytic sheaf $\mathfrak{F}(X)$ there exists a finite system of elements $f^i \in H^0(X, \mathfrak{F})$ such that the stalk $\mathfrak{F}_z$ at each point $z \in X$ is generated by the elements f^i as an $\mathfrak{O}_z$-module.*

Evidently Theorem (A) results from Theorem (A′). Conversely, if Theorems (A) and (B) are valid for the cube X, then, as was established in the proof of the Corollary 2 of Theorem 9.6, there exists a finite system of elements of the group $H^0(X, \mathfrak{F})$ generating the $\mathfrak{O}_z$-module $\mathfrak{F}_z$ at each point $z \in X$.

PROOF. The assertion of Theorems (A′) and (B) for the cube of null-dimension is obvious. Suppose that it is valid for the cube of topological dimension $r-1$; we must show that then the assertion of Theorem (A′) is true for the

topologically r-dimensional cube X.

We shall assume that the sheaf $\mathfrak{F}$ is defined in some neighborhood of the compactum X (see Theorem 9.1). The cube X may be regarded as the product of a closed interval I and an $(r-1)$-dimensional cube in the space R_{2n-1}. Let $X(t) \subset C^n$ be the cube of dimension $r-1$ obtained as the section of the cube X at the height $t \in I$ By Theorem (A′), which is true for the cube $X(t)$, there exists a finite set of sections from $H^0(X(t), \mathfrak{F})$ generating the stalk $\mathfrak{F}_z$ as an $\mathfrak{O}_z$-module at each point $z \in X$. Because of the compactness of the cube $X(t)$ it possesses a neighborhood to which these sections can be extended and where they continue to generate the stalk $\mathfrak{F}_z$ as an $\mathfrak{O}_z$-module. Moreover, the closed interval I can be decomposed into a finite system of closed intervals I_k, $k = 1, 2, \cdots$, with the following property: for each number k there exists over the union $\bigcup_{t \in I_k} X(t) = X_k$ a finite system of sections from $H^0(X_k, \mathfrak{F})$ generating the stalk $\mathfrak{F}_z$ at any point $z \in X_k$ as a $\mathfrak{O}_z$-module. We may suppose that the open intervals I_k do not overlap.

To prove Theorem (A′) for the cube X it is now sufficient to prove the following assertion (A″):

Let X' and X'' be the sets of points of the cube X for which $t \leq t_0$ and $t \geq t_0$, respectively. Assume that we are given two finite systems of sections $\phi_i' \in H^0(X', \mathfrak{F})$, $i = 1, \cdots, p$, and $\psi_j'' \in H^0(X'', \mathfrak{F})$, $j = 1, \cdots, q$, generating the $\mathfrak{O}_z$-modules $\mathfrak{F}_z$ respectively at points $z \in X'$ and $z \in X''$. Then there exists a finite system of sections $\chi_k \in H^0(X, \mathfrak{F})$ generating the $\mathfrak{O}_z$-module $\mathfrak{F}_z$ at all points $z \in X$.

It follows from Theorem 9.6 that the elements ϕ_i', as well as the elements ψ_j'', generate the group of sections $H^0(X' \cap X'', \mathfrak{F})$ over the ring of functions holomorphic on the cube $X' \cap X''$.

We introduce q elements $\psi_j' \in H^0(X', \mathfrak{F})$ and p elements $\phi_i'' \in H^0(X'', \mathfrak{F})$ by taking them all equal to zero. We now prove that there exists over the cube $X' \cap X''$ a holomorphic and invertible matrix M of order $p + q$ such that

$$M(\phi'', \psi'') = (\phi', \psi'). \tag{2.26}$$

Here (ϕ'', ψ'') and (ϕ', ψ') are column-matrices consisting of $p + q$ elements. This fact follows from the following algebraic lemma:

LEMMA 4. *Let A be a ring with a unit, let $\mathfrak{X}$ be a left unitary A-module,*[1]

1) A left A-module $\mathfrak{X}$ is said to be *unitary* if the unit element e of the ring A is neutral for composition with all elements of the module $\mathfrak{X}$ (i.e., $ex = x$ for all $x \in \mathfrak{X}$).

and let elements $u_i' \in \mathfrak{X}$, $i = 1, \cdots, p$ *and* $v_j'' \in \mathfrak{X}$, $j = 1, \cdots, q$ *be given. Suppose that* $v_j'' = \Sigma_i a_{ji} u_i'$; $u_i' = \Sigma_j b_{ij} v_j''$; $i = 1, \cdots, p$, $j = 1, \cdots, q$, *where* a_{ji}, $b_{ij} \in A$. *Then there exists a square matrix* M *of order* $p + q$ *consisting of elements of the ring* A *such that*

$$M(0, v'') = (u', 0). \tag{2.26_1}$$

PROOF OF THE LEMMA. Let $\mathfrak{X}' = \mathfrak{X}^p$ be an A-module consisting of systems $x' = \{x_i', i = 1, \cdots, p\}$, and let $\mathfrak{X}'' = \mathfrak{X}^q$ be an A-module consisting of systems $x'' = \{x_j'', j = 1, \cdots, q\}$, where all $x_i', x_j'' \in \mathfrak{X}$. Then the element $u' = \{u_1', \cdots, u_p'\} \in \mathfrak{X}'$ and the element $v'' = \{v_1'', \cdots, v_q''\} \in \mathfrak{X}''$. We consider

1) the elements $(0, v'')$, $(u', 0) \in \mathfrak{X}' + \mathfrak{X}''$, where the A-module $\mathfrak{X}' + \mathfrak{X}''$ is the direct sum of A-modules $\mathfrak{X}'$ and $\mathfrak{X}''$;

2) the mapping M: $\mathfrak{X}' + \mathfrak{X}'' \longrightarrow \mathfrak{X}' + \mathfrak{X}''$ defined by means of linear transformations a: $\mathfrak{X}' \longrightarrow \mathfrak{X}''$ and b: $\mathfrak{X}'' \longrightarrow \mathfrak{X}'$, where

$$a\{x_i'\} = \sum_i a_{ji} x_i'; \qquad b\{x_j''\} = \sum_j b_{ij} x_j''$$

by the formula

$$M(x', x'') = \big(b(a(x')) - b(x'') - x',\ a(x') - x''\big). \tag{2.27}$$

Evidently the mapping M may be represented in the form of the composition of two automorphisms. The first of them carries some element $(x', x'') \in \mathfrak{X}' + \mathfrak{X}''$ into the element $(x', a(x') - x'')$, and the second into the element $(b(x'') - x', x'')$. These mappings are automorphisms, since their squares reduce to the identity mapping. Furthermore, it is easy to see that the first of these automorphisms maps the element (u', v'') into the element $(u', 0)$ and the second into the element $(0, v'')$. Hence in view of formula (2.27) it follows that M is an automorphism and equation (2.26_1) is valid for it. Thus Lemma 4 is proved.

COMPLETION OF THE PROOF OF THEOREM (A$'$). By the lemma just proved there exists over the cube $X' \cap X''$ a holomorphic invertible matrix M satisfying the condition (2.26). Then, by Theorem 10.1, one can define in the parts X' and X'' of the cube X, respectively, holomorphic and invertible matrices M' and M'' such that $M'M'' = M$ at points $z \in X' \cap X''$. From relation (2.26) we further find that

$$M''(\varphi'', \psi'') = M'^{-1}(\varphi', \psi') \tag{2.28}$$

Here $M''(\phi'', \psi'')$ is a system consisting of $p + q$ elements of the group $H^0(X'', \mathfrak{F})$ and $M'^{-1}(\phi', \psi')$ is a system consisting of $p + q$ elements of the group $H^0(X', \mathfrak{F})$. Equation (2.28) shows that the functions constituting these

systems coincide in the cube $X' \cap X''$. As a result we obtain the system of $p + q$ sections of the sheaf $\mathfrak{F}$ over the cube $X' \cap X'' = X$ generating the $\mathfrak{O}_z$-module $\mathfrak{F}_z$ at every point $z \in X$. Thus the assertion (A$''$) and therewith Theorem (A$'$) are proved.

6. Proof of Theorem (B) for cubes in the space C^n. We assume, as in the preceding subsection, that the assertion of Theorem (B) holds for cubes in the space C^n of topological dimension smaller than r.

Let $X \subset C^n$ be a topologically r-dimensional cube, and let $\mathfrak{F}$ be a coherent analytic sheaf given over some neighborhood of the cube X. We have to prove that $H^q(X, \mathfrak{F}) = 0$ for $q \geq 1$.

By definition $H^q(X, \mathfrak{F})$ is the inductive limit of cohomology groups $H^q(\mathfrak{U}, \mathfrak{F})$ of open covering of the space X. The meaning of the equality $H^q(X, \mathfrak{F}) = 0$ is that for any covering $\mathfrak{U}$ of the space X and any element $\alpha \in H^q(\mathfrak{U}, \mathfrak{F})$ one can find a covering $\mathfrak{V} \prec \mathfrak{U}$, such that $\sigma(\mathfrak{U}, \mathfrak{V})\ \alpha = 0$ (see subsection 3 of the Introduction). This in its turn means: if $\gamma \in C^q(\mathfrak{U}, \mathfrak{F})$ is a cocycle representing the element α, while τ is a correspondence between indices of the coverings $\mathfrak{V}$ and $\mathfrak{U}$, then the cochain $\tilde{\tau}\gamma \in C^q(\mathfrak{V}, \mathfrak{F})$ is cohomologous to zero. In other words, there exists a cochain $\beta \in C^{q-1}(\mathfrak{V}, \mathfrak{F})$ such that $\delta\beta = \tilde{\tau}\gamma$ in the covering $\mathfrak{V}$.

In the sequel we usually preserve the notations α, γ, etc. for the cochains $\tilde{\tau}\alpha$, $\tilde{\tau}\gamma$, etc. (since this cannot lead to a misunderstanding). Then the last equation is written as $\delta\beta = \gamma$. Our aim is to prove that this equation actually holds.

Just as in the preceding subsection, we represent the compact cube X as the union of topologically $(r-1)$-dimensional cubes $X(t)$ where $t \in I$. For each value $t_0 \in I$ the covering $\mathfrak{U}$ generates an open covering $\mathfrak{U}(t_0)$ of the cube $X(t_0)$ and the cocycle $\gamma \in C^q(\mathfrak{U}, \mathfrak{F})$ generates a cocycle $\gamma(t_0) \in C^q(\mathfrak{U}(t_0), \mathfrak{F})$. By the hypothesis of the induction there exists a finite covering $\mathfrak{V}(t_0) \prec \mathfrak{U}(t_0)$ of the cube $X(t_0)$ and a cochain $\beta(t_0) \in C^{q-1}(\mathfrak{V}(t_0), \mathfrak{F})$ such that $\delta\beta(t_0) = \gamma(t_0)$ in the covering $\mathfrak{V}(t_0)$. We extend each element of the covering $\mathfrak{V}(t_0)$ to an open set in the cube X. We carry out this extension in order that as a result

1) we should obtain a covering $\tilde{\mathfrak{V}}$ of the union $X' = \bigcup_{t \in I'} X(t)$, where $I' \subset I$ is a closed interval refined in the open interval I and generated by the covering $\mathfrak{U}$;

2) there should exist cochains $\tilde{\beta} \in C^{q-1}(\tilde{\mathfrak{V}}, \mathfrak{F})$ and $\tilde{\gamma} \in C^q(\tilde{\mathfrak{V}}, \mathfrak{F})$ such that $\delta\tilde{\beta} = \tilde{\gamma}$, while the cochain $\tilde{\beta}$ generates the cochain $\beta(t_0) \in C^{q-1}(\mathfrak{V}(t_0), \mathfrak{F})$

and the cochain $\tilde{\gamma}$ generates the cochain $\gamma(t_0) \in C^q(\mathfrak{V}(t_0), \mathfrak{F})$;

3) each element of the covering thus obtained of the domain X' should be contained in some element of the covering $\mathfrak{U}$ of the cube X.

The covering constructed in this way may be extended by the addition of new elements to a covering of the entire cube X refined in the covering $\mathfrak{U}$. We may assume that there exists a closed interval $I'' \subset I$ containing the point t_0 such that the new elements of the covering have no common points with the union $\bigcup_{t \in I''} X(t)$. We extend the cochain $\tilde{\beta}$ to the whole of the new $(q-1)$-fold intersections by associating with it the zero elements of the corresponding groups.

We may suppose that such a construction is carried out for each closed interval I_k belonging to some finite covering $\{I_k, k \in K\}$ of the closed interval I. As a result we obtain for each number k a finite covering $\mathfrak{V}$ of the cube X refined in the covering $\mathfrak{U}$ and a cochain $\beta_k \in C^{q-1}(\mathfrak{V}, \mathfrak{F})$ such that the cochain $\delta\beta_k - \gamma$ in the covering $\mathfrak{V}$ induces the null cochain over the union $X_k = \bigcup_{t \in I_k} X(t)$.

Theorem (B) for the cube X will be proved if we establish the following assertion (B*):

Let X' and X'' be the sets of points of the cube X for which $t \leq t_0$ and $t \geq t_0$ respectively. Assume that there is a covering $\mathfrak{V}$ of the cube X and that there are given cochains $\beta_1, \beta_2 \in C^{q-1}(\mathfrak{V}, \mathfrak{F})$ and a cocycle $\gamma \in C^q(\mathfrak{V}, \mathfrak{F})$, while $\delta\beta_1 - \gamma = 0$ over X' and $\delta\beta_2 - \gamma = 0$ over X''. Then there exists (in the covering, which may be finer than $\mathfrak{V}$) a $(q-1)$-dimensional cochain β such that $\delta\beta = \gamma$ over the entire cube X.

First we assume that $q \geq 2$. The covering $\mathfrak{V}$ of the cube X generates a covering $\mathfrak{V}_0$ of the cube $X' \cap X'' = X(t_0)$, and the cochains β_1 and β_2 generate, respectively, cochains β_1^0 and β_2^0 over the cube $X(t_0)$. Evidently $\delta(\beta_1^0 - \beta_2^0) = 0$. In view of Theorem (B), which is true for the topologically $(r-1)$-dimensional cube, there exists a cochain $\gamma \in C^{q-2}(\mathfrak{V}_0, \mathfrak{F})$ such that $\delta\gamma = \beta_1^0 - \beta_2^0$. Reasoning in the same way as for the formation of the cochains β_k, we construct a covering $\mathfrak{V}'$ of the cube X refined in the covering $\mathfrak{V}$ and generating on the intersection $X' \cap X''$ the same covering $\mathfrak{V}_0$ as does the covering $\mathfrak{V}$, and a cochain $\alpha \in C^{q-2}(\mathfrak{V}', \mathfrak{F})$ such that $\delta\alpha = \beta_1 - \beta_2$ over some neighborhood of $X' \cap X''$. Consider the cochain

$$\beta = \begin{cases} \beta_1 & \text{over } X', \\ \beta_1 + \delta\alpha & \text{over } X''. \end{cases}$$

Evidently $\delta\beta = \gamma$ in the covering $\mathfrak{V}'$ of the cube X. Thus the assertion

(B^*) and therewith Theorem (B) for the cube X are proved for $q \geq 2$.

Now we take $q = 1$. Then, as in the previous case, we have

$$\delta(\beta_1^0 - \beta_2^0) = 0 \tag{2.29}$$

in the covering $\mathfrak{V}_0$ of the cube $X' \cap X''$. Since the cochain $\beta_1^0 - \beta_2^0$ is null-dimensional, relation (2.29) means that it is a section of the sheaf $\mathfrak{F}$ over the cube $X' \cap X''$ (see Theorem 0.2). Theorem (A$'$) holds for the cube X; therefore there exists a finite system of sections $\chi_i \in H^0(X, \mathfrak{F})$ generating the stalk $\mathfrak{F}_z$ as an $\mathfrak{O}_z$-module at any point $z \in X$, in particular, at any point $z \in X' \cap X''$. Theorem (A$'$) holds also for the cube $X' \cap X''$; therefore by Theorem 9.6 the sections χ_i generate the group $H^0(X' \cap X'', \mathfrak{F})$ as an $\mathfrak{O}_{X' \cap X''}$-module. In particular, for $z \in X' \cap X''$ the equality

$$\beta_1^0 - \beta_2^0 = \sum_i \lambda_i \chi_i$$

holds, where λ_i are functions holomorphic in some neighborhood of the compact cube $X' \cap X''$. We may assume that the cochain $\beta_1 - \beta_2$ is a cochain in the neighborhood of the cube $X' \cap X''$ and that therein

$$\beta_1 - \beta_2 = \sum_i \lambda_i \chi_i.$$

By the use of Cauchy's integral (as in the proof of Lemma 3) we express λ_i in the form $\lambda_i = \lambda_i' - \lambda_i''$, where λ_i' are functions holomorphic in a neighborhood U' of the compactum X', while λ_i'' are functions holomorphic in a neighborhood U'' of the compactum X''. Let $\mathfrak{V}' < \mathfrak{V}$ be a covering such that each element $V_i' \in \mathfrak{V}'$ intersecting with the compactum X' is contained in the neighborhood U' and each element $V_i' \in \mathfrak{V}'$ intersecting with the compactum X'' is contained in the neighborhood U''. The sections $\Sigma_i \lambda_i' \chi_i$ and $\Sigma_i \lambda_i'' \chi_i$ may be considered as null-dimensional cochains on the compactum X, which are equal to zero on all of those elements of the covering $\mathfrak{V}'$ that do not intersect respectively with the compacta X' and X''. Then the cochain $\beta \in C^0(\mathfrak{V}', \mathfrak{F})$, equal to $\beta_1 - \Sigma_i \lambda_i' \chi_i$ on the compactum X' and $\beta_2 - \Sigma_i \lambda_i'' \chi_i$ on the compactum X'', has the cocycle γ as its coboundary. Thus Theorem (B) is also proved in the case $q = 1$ for the cube $X \subset C^n$.

§11. PROOF OF THEOREMS (A) AND (B) FOR HOLOMORPHICALLY COMPLETE COMPLEX MANIFOLDS

1. Proof of Theorems (A) and (B) for certain compacta in complex manifolds.

LEMMA 1. *The compact polycylinder $\overline{D} \subset C^n$ always has a fundamental system of compact neighborhoods, each of which may be mapped biholomorphically*

onto a compact cube of the space C^n.

PROOF. The compactum $\overline{D}$ has a fundamental system of (open) neighborhoods, each of which is a polycylinder. Let $V = v_1 \times \cdots \times v_n$ be one of these neighborhoods. Since each of the disks v_i can be mapped conformally onto a square, there exists a biholomorphic mapping $f\colon V \longrightarrow Q$ where Q is an (open) cube in the space C^n. Lemma 1 will be proved if in Q we take a compact cube containing the image $f(\overline{D})$.

We note some consequences of the lemma just proved. From this lemma and Theorem 9.3 there result

THEOREM 11.1. *The compact polycylinder* $\overline{D} \subset C^n$ *has the property* (A).

THEOREM 11.2. *The compact polycylinder* $\overline{D} \subset C^n$ *has the property* (B).

Applying Theorem 9.3 again, we obtain the following proposition.

THEOREM 11.3. *Let* $\overline{D}$ *be a compact polycylinder, and let* E *be a complex manifold locally regularly and analytically imbedded in some neighborhood of the compactum* $\overline{D}$ *and closed in that neighborhood. Then the compactum* $\overline{D} \cap E$ *has the properties* (A) *and* (B) (*in the complex structure of the space* C^n).

Now we can prove

THEOREM 11.4. *If a complex manifold* $Z = Z^n$ *has the following properties*:

(a) *it is holomorphically separable*;

(b) *there exists at each point* $z \in Z$ *a system of functions holomorphic on the manifold* Z *and playing, in some neighborhood of the point* z, *the role of local coordinates*; *then any compactum* $K \subset Z$, *coinciding with its hull and convex with respect to the family of functions* $\mathfrak{O}_Z$ *has the properties* (A) *and* (B).

REMARK. Holomorphically complete complex manifolds have the properties (a) and (b) stated in the present theorem (see §9.3). Therefore Theorem 11.4 establishes that *holomorphically convex compacta in holomorphically complete manifolds have the properties* (A) *and* (B).

The proof of Theorem 11.4 is based on the following lemma:

LEMMA 2. *The compactum* K *from Theorem* 11.4 *has a fundamental system of compact neighborhoods, each of which represents the intersection* $\overline{D} \cap E$ *considered in Theorem* 11.3.

PROOF OF THE LEMMA. We take a covering $\mathfrak{V}$ of the manifold Z satisfying the condition (b). We may suppose that elements $v_1, \cdots, v_n$ of that covering

cover the compactum K. Let $f_{i1}, \cdots, f_{in}$ be functions holomorphic on the entire manifold Z, playing the part of local coordinates in the element $\tilde{v}_i \in \mathfrak{V}$. Set $U = \bigcup_{i=1}^m v_i$. We denote by $\tilde{K}$ a compactum such that $U \supset \tilde{H} \supset V \supset K$. Here V is some (open) neighborhood of the compactum K.

We form the product $\tilde{K} \times \tilde{K}$ and consider a neighborhood ∇ of its "diagonal" (concerning this concept see §3.5, Chapter I) defined by the condition: if a point $w(z^1, z^2) \in \nabla$, then both points $z^1, z^2 \in \tilde{K}$ belong to one and the same element $v_i \in \mathfrak{V}$, $i = 1, \cdots, m$. Next we construct the compactum $\mathfrak{K} = \tilde{K} \times \tilde{K} \backslash \nabla$. For each point $w(z^1, z^2) \in \mathfrak{K}$ one can find a function $f_w(z) \in \mathfrak{O}_Z$ such that $f_w(z^1) \neq f_w(z^2)$. Evidently this function f_w will also separate the points $z_1^1, z_1^2 \in Z$ for points $w_1(z_1^1, z_1^2) \in \mathfrak{U}$, where $\mathfrak{U} \subset \mathfrak{K}$ is some neighborhood of the point w. For points $w_1 \in \mathfrak{U}$ one may put $f_{w_1} = f_w$. Hence, from the compactness of the set $\mathfrak{K}$ it follows that the functions f_w, $w \in \mathfrak{K}$, can be so chosen that among them only a finite set of functions $f_1, \cdots, f_s$ are distinct. Then the functions

$$f_1, \cdots, f_s, f_{i1}, \cdots, f_{in}, \quad i = 1, \cdots, m, \tag{2.30}$$

holomorphic on the entire manifold Z, separate every point of the compactum $\tilde{K}$. Indeed, let points $z^1, z^2 \in \tilde{K}$ and $z^1 \neq z^2$. Then if these points belong to one and the same element $v_i \in \mathfrak{V}$, they are separated at least by one of the local coordinates, i.e., one of the functions $f_{i1}, \cdots, f_{in}$. If the points z^1 and z^2 do not belong to the same element v_i, then $w(z^1, z^2) \in \mathfrak{K}$ and these points are separated by one of the functions $f_1, \cdots, f_s$.

Without loss of generality we may assume that all the functions f_{ik} and f_r on the compactum $\tilde{K}$ are less than unity in modulus.

Now let $\tilde{K}_\nu$, $\nu = 1, 2, \cdots$, $\tilde{K} \subset \tilde{K}_\nu \subset K$, be a fundamental system of compact neighborhoods of the compactum K; let U_ν, $\nu = 1, 2, \cdots$, be open sets such that $\tilde{K}_\nu \subset U_\nu \subset \overline{U}_\nu \subset K$. Then, in view of the convexity of the compactum K with respect to functions holomorphic on the entire manifold Z, one can find at each point $\zeta \in \tilde{K}_\nu \backslash U_\nu$ a function $g_\zeta(z) \in \mathfrak{O}_Z$ such that $|g_\zeta(\zeta)| > 1$ and $|g_\zeta(z)| < 1$ for $z \in K$. The set $\tilde{K}_\nu \backslash U_\nu$ is compact. Therefore there exists a finite system of functions $g_j^\nu(z) \in \mathfrak{O}_Z$, $j = 1, \cdots, p_\nu$, $|g_j^\nu(z)| < 1$ for $z \in K$, with the following property: at each point $\zeta \in \tilde{K}_\nu \backslash U_\nu$ one can find a number l such that $|g_l^\nu(\zeta)| > 1$. Take an open set V_ν such that $\tilde{K}_\nu \supset \overline{V}_\nu \supset V_\nu \supset \overline{U}_\nu$ and consider a mapping $\phi_\nu \colon V_\nu \longrightarrow C^N$, where $N = mn + s + p_\nu$, defined by means of the functions

$$\left.\begin{array}{ll} f_1, \ldots, f_s, g_j^{\nu}, & j=1, \ldots, p_{\nu}, \\ f_{i1}, \ldots, f_{in}, & i=1, \ldots, m. \end{array}\right\} \qquad (2.31)$$

We show that this mapping is a regular analytic imbedding (see §14.1, Chapter III, (I)), i.e., that

1) it is holomorphic;

2) under this mapping different points go over into different points;

3) the rank of the Jacobian for the system of functions (2.31) takes on its maximal value at all points of the set V_ν.

Indeed, the mapping has the property 1) since all of the functions (2.31) are holomorphic. It possesses the property 2) since the functions (2.30) separate the points of the compactum $\tilde{K}$. Finally, the presence of the property 3) results from the fact that at each point $z \in V_\nu$ functions $f_{i1}, \cdots, f_{im}$ with a certain number i play the role of local coordinates.

Evidently $\phi_\nu(K) \subset Q^N$, where Q^N is the unit polycylinder with its center at the origin of coordinates of the space C^N, and $\phi_\nu(V_\nu \setminus U_\nu) \subset C^N \setminus \bar{Q}^N$. We set $\phi_\nu^{-1}(\bar{Q}^N) = K_\nu$. Then $U_\nu \subset K_\nu \subset K$. The compacta K_ν have all the required properties. The lemma is proved.

PROOF OF THEOREM 11.4. The proof is reduced to an application of the lemma just proved together with Theorems 9.3 and 11.3.

2. Auxiliary propositions.

THEOREM 11.5. *If a complex manifold Z and a compactum $K \subset Z$ have the properties stated in the preceding theorem, then any function holomorphic on the compactum K can be approximated uniformly on K by functions holomorphic on the entire manifold Z.*

PROOF. Just as we did in the proof of Lemma 2, we construct a regular and analytic imbedding ϕ of a neighborhood V of the compactum K into a closed unit polycylinder E of the space C^N, defined by means of functions belonging to $\mathfrak{O}_Z$. By Theorem 9.7 every function holomorphic on the image $\phi(V)$ is a restriction on this image of a function holomorhic in the polycylinder E. Our assertion follows from the fact that every function holomorphic in the polycylinder E can be uniformly approximated by polynomials (on this question see Example 1) in §3.2, Chapter I).

Let Z be a holomorphically complete complex manifold. It is a union of a countable set of compacta $Z = \bigcup_{i=1}^{\infty} K_i$ (see §9.3). We may assume that the compacta K_i are strictly imbedded one within another: $K_i \Subset K_{i+1}$, $i = 1, 2, \cdots$.

If for a certain sequence of points $\{x_i \in Z,\ i = 1, 2, \cdots\}$ each point $x_i \notin K_i$, then this sequence diverges on the manifold Z. Therefore if the compactum $K \Subset Z$, then $K \Subset K_i$, beginning with some sufficiently large number i. We set $\widetilde{\mathfrak{K}}_1 = \widehat{K}_1$ (here, and in what follows, $\widehat{K}_i$ is the hull of the compactum K_i, convex with respect to the function family $\widehat{\mathfrak{O}}_Z$). Then there exists a compactum $K_{i_1} \Supset \mathfrak{K}_1$. We set $\mathfrak{K}_2 = \widehat{K}_{i_1}$ and so forth. As a result we construct a sequence of compacta $\mathfrak{K}_i$, $i = 1, 2, \cdots$, $\mathfrak{K}_i \Subset \mathfrak{K}_{i+1}$; each of them coincides with its hull, which is convex with respect to the function family $\mathfrak{O}_Z$; the union of these compacta makes up the entire manifold Z.

In view of Theorem 11.4 each of the compacta $\mathfrak{K}_i$ has the properties (A) and (B); by Theorem 11.5 any function holomorphic in some neighborhood of the compactum $\mathfrak{K}_i$ can be represented on this compactum as a limit of a uniformly convergent sequence of functions belonging to the family $\mathfrak{O}_Z$.

Let $\mathfrak{F}(Z)$ be a coherent analytic sheaf. Then one can find for each compactum $\mathfrak{K}_i$ a system of elements $f_i^k \in H^0(\mathfrak{K}_i, \mathfrak{F})$, $k = 1, \cdots, \kappa_i$, generating the group $H^0(\mathfrak{K}_i, \mathfrak{F})$ as an $\mathfrak{O}_{\mathfrak{K}_i}$-module. This follows from Theorem (A) (valid for the compactum $\mathfrak{K}_i$) and Theorem 9.6.

We agree to call the following quantity the *seminorm* of an element $s \in H^0(\mathfrak{K}_i, \mathfrak{F})$:

$$\|s\|_i = \inf_{\{c_k\}} \max_{z \in \mathfrak{K}_i} \left(|c_1(z)|, \ \ldots, \ |c_{\kappa_i}(z)| \right), \tag{2.32}$$

where $c_k(z) \in \mathfrak{O}_{\mathfrak{K}_i}$ and

$$\sum_k c_k f_i^k = s. \tag{2.33}$$

The greatest lower bound in formula (2.32) is to be taken with respect to the system of functions $\{c_k\}$ for which equation (2.33) holds. This seminorm naturally defines a topology in the set of elements $H^0(\mathfrak{K}_i, \mathfrak{F})$.[1)] This topology is not Hausdorff: generally speaking, it does not follow from the equality $\| s \|_i = 0$ that $s = 0$. Only the following weaker assertion holds:

THEOREM 11.6. *If an element* $s \in H^0(\mathfrak{K}_{i+1}, \mathfrak{F})$ *and* $\| s \|_{i+1} = 0$, *then the restriction of the element* s *on the compactum* $\mathfrak{K}_i$ *is equal to zero.*

PROOF. By the hypothesis of the theorem we have

$$s = \sum_k c_k f_{i+1}^k \quad \text{for} \quad z \in \mathfrak{K}_{i+1}, \tag{2.34}$$

1) It can be shown that this topology does not depend on the choice of elements f_i^1, $f_i^2, \cdots$. However we shall not use this fact later.

where the functions $c_k \in \mathfrak{O}_{\mathfrak{R}_{i+1}}$. Since $\| s \|_{i+1} = 0$, one can find for any number $N > 0$ a function $b_k^{(N)} \in \mathfrak{O}_{\mathfrak{R}_{i+1}}$, $| b_k^{(N)} | < 2^{-N}$ for $z \in \mathfrak{R}_{i+1}$, such that

$$s = \sum_k b_k^{(N)} f_{i+1}^k \quad \text{for} \quad z \in \mathfrak{R}_{i+1} \tag{2.35}$$

From equations (2.34) and (2.35) it follows that for $z \in \mathfrak{R}_{i+1}$,

$$\sum_k c_k^{(N)} f_{i+1}^k = 0, \tag{2.36}$$

where the quantities $c_k^{(N)} = c_k - b_k^{(N)}$ for all indices k tend uniformly to zero as $N \longrightarrow \infty$ on the entire compactum $\mathfrak{R}_{i+1}$. Let z be an interior point of the compactum $\mathfrak{R}_{i+1}$. In the stalk $\mathfrak{O}_z^\nu$, where $\nu = \kappa_{i+1}$, we select an $\mathfrak{O}_z$-submodule consisting of those systems $(\zeta_1, \zeta_2, \cdots, \zeta_\nu) \in \mathfrak{O}_z^\nu$ for which $\Sigma_k \zeta_k f_{i+1}^k(z) = 0$. In view of (2.36) the systems $(c_1^{(N)}, \cdots, c_\nu^{(N)})$ belong to this submodule. Then, by Theorem 8.2, the system $(c_1, \cdots, c_\nu)$ also belongs to it. Thus at every interior point z of the compactum $\mathfrak{R}_{i+1}$

$$\sum_k c_k f_{i+1}^k = s = 0.$$

Hence our assertion follows since $\mathfrak{R}_i \Subset \mathfrak{R}_{i+1}$.

THEOREM 11.7. *The natural homomorphism* ϕ: $H^0(\mathfrak{R}_{i+1}, \mathfrak{F}) \longrightarrow H^0(\mathfrak{R}_i, \mathfrak{F})$ *is continuous in the sense of the seminorm.*

PROOF. Let $\{f_i^s\}$ and $\{f_{i+1}^s\}$ be finite systems of elements generating the groups $H^0(\mathfrak{R}_i, \mathfrak{F})$ and $H^0(\mathfrak{R}_{i+1}, \mathfrak{F})$ in the lattice of corresponding modules (generally speaking, these systems of elements are given independently of each other). Then the homomorphism ϕ will be given by the equation

$$f_{i+1}^j = \sum_{k=1}^{i} \lambda_k^j f_i^k, \qquad j = 1, \ldots, \varkappa_{i+1},$$

where $\lambda_k^j \in \mathfrak{O}_{\mathfrak{R}_{i+1}}$. Since $\mathfrak{R}_{i+1}$ is a compactum, the moduli $| \lambda_k^j |$ are bounded on $\mathfrak{R}_{i+1}$. Suppose for example that $| \lambda_k^j | \leq \alpha_i$ for all values of indices k and j and for $z \in \mathfrak{R}_{i+1}$. Then, so long as the image of the element $s = \Sigma_j c_j f_{i+1}^j \in H^0(\mathfrak{R}_{i+1}, \mathfrak{F})$ in the group $H^0(\mathfrak{R}_i, \mathfrak{F})$ is the element $\Sigma_k \Sigma_j c_j \lambda_k^j f_i^k$, we have

$$\| s \|_i \leq \alpha_i \| s \|_{i+1}. \tag{2.37}$$

Hence our assertion follows.

THEOREM 11.8. *Let* $s_p \in H^0(\mathfrak{R}_{i+1}, \mathfrak{F})$, $p = 1, 2, \cdots$, *be a sequence for which the series* $\Sigma_p \| s_p \|_{i+1}$ *converges. Then there exists an element* $s \in H^0(\mathfrak{R}_i, \mathfrak{F})$ *such that* $\lim_{q \to \infty} \| s - \Sigma_{p \leq q} s_p \|_i = 0$.

PROOF. In our case $s_p = \Sigma_j c_j^p f_{i+1}^j$, $p = 1, 2, \cdots$, where $|c_j^p| \leq 2 \| s_p \|_{i+1}$ for $z \in \mathfrak{R}_{i+1}$. For each number i the series $\Sigma_j c_j^p f_{i+1}^j$ consists of holomorphic functions and converges on the compactum $\mathfrak{R}_{i+1}$. Its sum is holomorphic on the compactum $\mathfrak{R}_i$. The element $s = \Sigma_j c_j f_{i+1}^j \in H^0(\mathfrak{R}_i, \mathfrak{F})$ satisfies the requirements of the theorem.

REMARK. The element s we have constructed is the "sum" of the series $\Sigma_p s_p$ in the sense of the seminorm being considered. This series may have some similar sums belonging to the group $H^0(\mathfrak{R}_i, \mathfrak{F})$. However, if s and s' are two such sums, then $\| s - s' \|_i = 0$ and therefore this series only has a sum in the group $H^0(\mathfrak{R}_{i-1}, \mathfrak{F})$.

THEOREM 11.9. *For every element $s \in H^0(\mathfrak{R}_i, \mathfrak{F})$ and a number $t > 0$ one can find an element $s' \in H^0(\mathfrak{R}_{i+1}, \mathfrak{F})$ such that $\| s - s' \|_i \leq t$.*

PROOF. Let elements f_{i+1}^j generate the stalk $\mathfrak{F}_z$ in the structure of the $\mathfrak{O}_z$-module at all points $z \in \mathfrak{R}_i$. Then by Theorem 9.6 they generate the group $H^0(\mathfrak{R}_i, \mathfrak{F})$ as an $\mathfrak{O}_{\mathfrak{R}_i}$-module. Consequently

$$s = \sum_j a_j f_{i+1}^j,$$

where the functions $a_j \in \mathfrak{O}_{\mathfrak{R}_i}$. By Theorem 11.5 there exist functions $b_j \in \mathfrak{O}_Z$ such that $| b_j - a_j | < t$ for $z \in \mathfrak{R}_i$. The element

$$s' = \sum_j b_j f_{i+1}^j$$

satisfies the condition of the theorem.

THEOREM 11.10. *For every element $s \in H^0(\mathfrak{R}_i, \mathfrak{F})$ and a number $t > 0$ one can find an element $s' \in H^0(Z, \mathfrak{F})$ such that $\| s - s' \|_i \leq t$.*

PROOF. By Theorem 11.9 there exists an element $s_{i+1} \in H^0(\mathfrak{R}_{i+1}, \mathfrak{F})$ such that $\| s_{i+1} - s \|_i \leq t/2$. Furthermore, by the same theorem, there exists an element $s_{i+2} \in H^0(\mathfrak{R}_{i+2}, \mathfrak{F})$ such that $\| s_{i+2} - s_{i+1} \|_{i+1} \leq t/4\alpha_i$ (the quantity α_i has been defined in the course of proving Theorem 11.7). Reasoning in a similar way we define a sequence of elements $s_{i+p} \in H^0(\mathfrak{R}_{i+p}, \mathfrak{F})$, where $s_i = s$ and $p = 0, 1, \cdots$, satisfying the inequality

$$\| s_{i+p+1} - s_{i+p} \|_{i+p} \leqslant 2^{-(p+1)} t (\alpha_i \alpha_{i+1} \ldots \alpha_{i+p-1})^{-1}. \tag{2.38}$$

From inequalities (2.37) and (2.38) it further follows that

$$\| s_{i+q+p+1} - s_{i+q+p} \|_{i+q} \leqslant 2^{-(p+q+1)} t (\alpha_i \ldots \alpha_{i+q-1})^{-1}, \tag{2.39}$$

where $q > 0$ is an arbitrary natural number.

Hence it follows that the series $\Sigma_p (s_{i+q+p+1} - s_{i+q+p})$, consisting of elements of the group $H^0(\mathfrak{K}_{i+q}, \mathfrak{F})$, has a unique sum belonging to the group $H^0(\mathfrak{K}_{i+q-2}, \mathfrak{F})$ in the sense of the remark following Theorem 11.8. We denote this sum by σ_{i+q-2}. Let

$$s'_{i+q-2} = (\sigma_{i+q-2} + s_{i+q+1}) \in H^0(\mathfrak{K}_{i+q-2}, \mathfrak{F}).$$

It is easy to see that the restriction of the element $s'_{i+q-1} \in H^0(\mathfrak{K}_{i+q-1}, \mathfrak{F})$ in the compactum $\mathfrak{K}_{i+q-2}$ is the element s'_{i+q-2}. Therefore the collection of these elements s'_k defines a section $s' \in H^0(Z, \mathfrak{F})$. This section s' satisfies the conditions of the theorem.

3. **Completion of the proof of Theorems (A) and (B) for holomorphically complete complex manifolds.**

PROOF OF THEOREM (A). Let Z be a holomorphically complete complex manifold, let $\mathfrak{K}_1 \Subset \mathfrak{K}_2 \Subset \cdots$ be the sequence of compacta defined in the preceding subsection, let $\mathfrak{F}(Z)$ be a coherent analytic sheaf, and let a point $z \in Z$. We must prove that the group $H^0(Z, \mathfrak{F})$ generates each stalk $\mathfrak{F}_z$ as an $\mathfrak{O}_z$-module.

Let ν be a natural number such that the point z belongs to the interior of the compactum $\mathfrak{K}_\nu$; we suppose that elements $f^1_\nu, \cdots, f^p_\nu \in H^0(\mathfrak{K}_\nu, \mathfrak{F})$ generate this group as an $\mathfrak{O}_{\mathfrak{K}_\nu}$-module. Let M be a submodule of the $\mathfrak{O}_z$-module $\mathfrak{O}^p_z$ formed by systems $(a_1, \cdots, a_p) \in \mathfrak{O}^p_z$ with the following property: $(a_1, \cdots, a_p) \in M$ if and only if the element $\Sigma_k a_k f^k_\nu$ belongs to the submodule of the $\mathfrak{O}_z$-module $\mathfrak{F}_z$ generated by the sections from the group $H^0(Z, \mathfrak{F})$. We must establish that $M = \mathfrak{O}^p_z$.

In view of Theorem 8.2 this will be established if we show that each element $(b_1, \cdots, b_p) \in \mathfrak{O}^p_z$ is a uniformly attained limit of a sequence of elements $(a^r_1, \cdots \cdots, a^r_p) \in M$, $r = 1, 2, \cdots$. We may assume without loss of generality that all germs $b_1, \cdots, b_p$ are equal to zero except for one which is equal to unity. In other words, it suffices to prove that any one of the elements $f^1_\nu, \cdots, f^p_\nu$ is a limit, in the sense of the seminorm $\| \cdots \|_\nu$, of elements generated in the group $H^0(\mathfrak{K}_\nu, \mathfrak{F})$ by the sections of the sheaf $\mathfrak{F}(Z)$ over the entire manifold Z. However, this follows immediately from Theorem 11.10.

Thus Theorem 9.8 (H. Cartan's Theorem (A)) is completely proved.

PROOF OF THEOREM (B). It is necessary to prove that $H^q(Z, \mathfrak{F}) = 0$ for any $q \geq 1$. In other words, we must prove that if $\mathfrak{U}$ is a covering of the manifold Z and a cocycle $\gamma \in C^q(\mathfrak{U}, \mathfrak{F})$, then there exists a covering $\mathfrak{V} \prec \mathfrak{U}$ and a cochain $\beta \in C^{q-1}(\mathfrak{U}, \mathfrak{F})$ such that $\delta\beta = \gamma$ in the covering $\mathfrak{V}$. It is known that for each

compactum $\mathfrak{K}_i$ the equality $H^q(\mathfrak{K}_i, \mathfrak{F}) = 0$ holds; accordingly there exists a covering $\mathfrak{V}_i$ of the manifold Z and a cochain $\beta_i \in C^{q-1}(\mathfrak{V}_i, \mathfrak{F})$ such that $\delta\beta_i = \gamma$ in the covering $\mathfrak{V}_i$ on the compactum $\mathfrak{K}_i$.

We may suppose that $\mathfrak{V}_i$, for any i, represents one and the same at most countable covering $\mathfrak{V}$ of the manifold Z. If this was not so initially, we may take as $\mathfrak{V}$ the union of the coverings $\mathfrak{V}_i$. By $\mathfrak{V} \cap \mathfrak{K}_i$ we further denote the covering of the compactum $\mathfrak{K}_i$ generated by the covering $\mathfrak{V}$ of the manifold Z. The cochain $\beta_{i+1} - \beta_i \in C^{q-1}(\mathfrak{V} \cap \mathfrak{K}_i, \mathfrak{F})$ is a cocycle. Consider two cases separately.

1) $q \geq 2$. Then there exist a covering $\mathfrak{W} \prec \mathfrak{V}$ and cochains $\alpha_i \in C^{q-2}(\mathfrak{W}, \mathfrak{F})$, $i = 1, 2, \cdots$, such that $\delta\alpha_i = \beta_{i+1} - \beta_i$ over the compactum $\mathfrak{K}_i$ in the covering $\mathfrak{W} \cap \mathfrak{K}_i$. It is readily seen that $\delta(\beta_{i+1} - \delta\alpha_i) = \gamma$ over the compactum $\mathfrak{K}_i$. The cochains $\beta_1 \in C^{q-1}(\mathfrak{W} \cap \mathfrak{K}_1, \mathfrak{F})$, $\beta_2 - \delta\alpha_1 \in C^{q-1}(\mathfrak{W} \cap \mathfrak{K}_2, \mathfrak{F}), \cdots, \beta_{i+1} - \delta\alpha_1 - \cdots - \delta\alpha_i \in C^{q-1}(\mathfrak{W} \cap \mathfrak{K}_{i+1}, \mathfrak{F})$ are generated by one another. All together they define a cochain $\beta \in C^{q-1}(\mathfrak{W}, \mathfrak{F})$ such that $\delta\beta = \gamma$ in the covering $\mathfrak{W}$.

2) $q = 1$. In this case the cochain $\beta_{i+1} - \beta_i = s \in H^0(\mathfrak{K}_i, \mathfrak{F})$. Then by Theorem 11.9 there exists an element $s' \in H^0(\mathfrak{K}_{i+1}, \mathfrak{F})$ such that $\| s' - s \|_i \leq t$ (where the number t can be chosen arbitrarily small). In other words, one can change the cochain β_{i+1} (without changing its coboundary $\delta\beta_{i+1}$) in such a way that the seminorm $\| \beta_{i+1} - \beta_i \|_i$ turns out to be arbitrarily small. Just as in the proof of Theorem 11.10, we choose the sections $\beta_{i+1}, \beta_{i+2}, \cdots$ over the corresponding compacta in such a way that the series $\Sigma_p (\beta_{i+r+p+1} - \beta_{i+r+p})$ will have as its "sum" (in the sense of the seminorm) the section σ_{i+r-2} over $\mathfrak{K}_{i+r-2}$. The cochains $\beta'_{i+r-2} = \sigma_{i+r-2} + \beta_{i+r+1}$ are extensions of one another (to $\mathfrak{K}_i$) and together they form the cochain $\beta' \in H^0(Z, \mathfrak{F})$. Since σ_{i+r-2} is a section over the compactum $\mathfrak{K}_{i+r-2}$, we have $\delta\sigma_{i+r-2} = 0$ and $\delta\beta'_{i+r-2} = \delta\beta_{i+r+1} = \gamma$ over the compactum $\mathfrak{K}_{i+r-2}$. Therefore $\delta\beta' = \gamma$.

Thus Theorem 9.9 (H. Cartan's Theorem (B)) is completely proved.

§12. SOLUTION OF FUNDAMENTAL PROBLEMS FOR HOLOMORPHICALLY COMPLETE COMPLEX MANIFOLDS [1]

1. Cousin's first problem for a complex manifold Z is as follows (see §7.1): It is required to establish whether or not the homomorphism $\pi^*: H^0(Z, \mathfrak{M}^1) \to H^0(Z, \mathfrak{H})$ generated by the homomorphism $\pi: \mathfrak{M}^1(Z) \to \mathfrak{H}(Z)$ is an epimorphism.

1) In our exposition we follow H. Cartan [5], Report XX (presented by J.-P. Serre), H. Cartan [4], (I) and Serre [1,2].

Here $\mathfrak{M}^1(Z)$ is a sheaf of abelian groups (with respect to addition) of germs of meromorphic functions and $\mathfrak{H}(Z) = \mathfrak{M}^1(Z)/\mathfrak{O}(Z)$ is a sheaf of principal parts of germs of meromorphic functions over the manifold Z.

We consider the exact sequence of cohomology groups

$$H^0(Z, \mathfrak{M}^1) \xrightarrow{\pi^*} H^0(Z, \mathfrak{H}) \xrightarrow{\mathfrak{g}} H^1(Z, \mathfrak{O}) \tag{2.40}$$

(where $\mathfrak{g}$ is a so-called Bockstein's homomorphism; see subsection 4 of the Introduction), generated by the exact sequence of sheaves

$$0 \to \mathfrak{O}(Z) \to \mathfrak{M}^1(Z) \to \mathfrak{H}(Z) \to 0.$$

In order that the homomorphism π^* should be an epimorphism, it is necessary and sufficient, because of the exactness of the sequence (2.40), that the kernel of the homomorphism $\mathfrak{g}$: $H^0(Z, \mathfrak{H}) \to H^1(Z, \mathfrak{O})$ should coincide with the entire group $H^0(Z, \mathfrak{H})$, i.e., that the homomorphism $\mathfrak{g}$ should be trivial. Thus we have the

THEOREM 12.1. *In order that Cousin's first problem should be solvable on a complex manifold Z, it is necessary and sufficient that the homomorphism $\mathfrak{g}$: $H^0(Z, \mathfrak{H}) \to H^1(Z, \mathfrak{O})$ should be trivial.*

Hence there follows the

THEOREM 12.2. *On a holomorphically complete complex manifold Cousin's first problem is always solvable.*

PROOF. For the holomorphically complete complex manifold Z the group $H^1(Z, \mathfrak{O}) = 0$ by Theorem 9.9 (H. Cartan's Theorem (B)). Therefore Theorem 12.2 follows from Theorem 12.1.

As was remarked before (see §18.4, Chapter III, (I)), every interiorly-nonbranching domain of holomorphy over the space C^n is a holomorphically complete complex manifold.

Therefore Theorem 12.2 is a generalization of Oka's Theorem 7.1 stating that Cousin's first problem is solvable for any domain of holomorphy of the space C^n.

We note that Cousin's first problem is not only solvable on a holomorphically complete complex manifold; for example, it is solvable for the complex projective space P^n. This last result follows from Theorem 12.1 and the equality $H^q(P^n, \mathfrak{O}) = 0$, valid for all values $q > 0$, in spite of the fact that the space P^n is not a holomorphically complete complex manifold. The last equality may be obtained as a consequence of Dolbeault's results [1] (see H. Cartan [4], (I)).

2. Cousin's second problem for a complex manifold Z is as follows (see §7.2): It is required to establish whether or not the homomorphism $\pi^*: H^0(Z, \mathfrak{M}^2) \longrightarrow H^0(Z, \mathfrak{D})$ generated by the homomorphism π: $\mathfrak{M}^2(Z) \longrightarrow \mathfrak{D}(Z)$ is an epimorphism. Here $\mathfrak{M}^2(Z)$ is the sheaf of abelian groups (with respect to multiplication) of germs of meromorphic functions with the exception of the identically zero function, and $\mathfrak{D}(Z)$ is the sheaf of germs of divisors over the manifold Z.

We consider, as in Cousin's first problem, the exact sequence of cohomology groups

$$H^0(Z, \mathfrak{M}^2) \xrightarrow{\pi^*} H^0(Z, \mathfrak{D}) \xrightarrow{\mathfrak{g}} H^1(Z, \mathfrak{J}),$$

where $\mathfrak{J}(Z)$ is the sheaf of germs of invertible holomorphic functions over the manifold Z (see §7.2) generated by the exact sequence of sheaves

$$0 \to \mathfrak{J}(Z) \to \mathfrak{M}^2(Z) \to \mathfrak{D}(Z) \to 0.$$

As a result, by repeating the argument of the preceding subsection, we obtain the

THEOREM 12.3. *In order that Cousin's second problem should be always solvable on a complex manifold Z, it is necessary and sufficient that the homomorphism $\mathfrak{g}$: $H^0(Z, \mathfrak{D}) \longrightarrow H^1(Z, \mathfrak{J})$ should be trivial.*

The analogy with the solution of Cousin's first problem stops at this point. The sheaf $\mathfrak{J}(Z)$ is not coherent and therefore one cannot directly apply Theorem (B).

First of all we prove the following auxiliary proposition.

THEOREM 12.4. *On the holomorphically complete complex manifold Z one can find, for any element $\Phi \in H^1(Z, \mathfrak{J})$, a positive divisor $D \in H^0(Z, \mathfrak{D})$ such that $\mathfrak{g}(D) = \Phi$.*

PROOF. The cohomology group with coefficients in a sheaf is a direct limit of cohomology groups of coverings. Therefore one can find a sufficiently small covering $\mathfrak{U} = \{U_i,\ i \in I\}$ such that the element $\Phi \in H^1(Z, \mathfrak{J})$ turns out to be the image of some element $\widetilde{\Phi} \in H^1(\mathfrak{U}, \mathfrak{J})$ under the homomorphism $\phi_{\mathfrak{U}}$ (see subsection 3 of the Introduction). Consider a cocycle from the group $C^1(\mathfrak{U}, \mathfrak{J})$ belonging to the class $\widetilde{\Phi}$. As a one-dimensional cochain this cocycle associates with each pair of indices $i, j \in I$ (where $i \neq j$ and $U_i \cap U_j \neq \emptyset$) the section of the sheaf $\mathfrak{J}(Z)$ over $U_i \cap U_j$, i.e., the invertible holomorphic function $f_{ij} \in \mathfrak{J}_{U_i \cap U_j}$. The coboundary of this cochain is a two-dimensional cochain; it associates with each triad of indices $i, j, k \in I$ (these indices are pairwise different;

$U_i \cap U_j \cap U_k \neq \emptyset$) the function $f_{ij}f_{jk}f_{ki} = f_{ij}f_{jk}f_{ik}^{-1}$ (in the sheaf $\mathfrak{F}(Z)$ the stalk is an abelian group of germs of invertible holomorphic functions with respect to multiplication; therefore we correspondingly write the element of the group multiplicatively, not additively as we did in the definition of the coboundary in subsection 3 of the Introduction). Since this cochain is a cocycle, $f_{ij}f_{jk}f_{ik}^{-1} = 1$ (we have written unity, not to zero, since the group is multiplicative) or $f_{ij}f_{jk} = f_{ik}$.

Over the intersection $U_i \cap U_j$ we consider the isomorphic mapping r_{ij}: $\mathfrak{O}(U_i) \longrightarrow \mathfrak{O}(U_j)$ defined by the formula $r_{ij}(\phi) = f_{ij}\phi$, where ϕ is a germ of holomorphic functions from the sheaf $\mathfrak{O}(U_j)$, while $f_{ij}\phi$ is the corresponding germ of holomorphic functions from the sheaf $\mathfrak{O}(U_j)$. The isomorphism r_{ij} has the transitivity property, i.e., $r_{ij}r_{jk} = r_{ik}$. Therefore, by identifying the sheaves $\mathfrak{O}(U_i)$ and $\mathfrak{O}(U_j)$ by the isomorphism r_{ij} in the intersection $U_i \cap U_j$ (i.e., regarding as identical the corresponding germs of holomorphic functions) , we define a certain sheaf $\mathfrak{P}(Z)$. This sheaf is locally isomorphic to the sheaf $\mathfrak{O}(Z)$ and thus it is coherent. By Theorem (A) (Theorem 9.8) there exists a nonzero section $p(z) \in \mathfrak{P}_Z$. It defines over each domain U_i an invertible holomorphic function p_i, for which the relation $p_i = f_{ij}p_j$ holds over every intersection $U_i \cap U_j$. Since f_{ij} are invertible holomorphic functions, the section $p(z)$ defines a certain divisor $D \in H^0(Z, \mathfrak{D})$. This divisor is positive, and so all the functions p_i are holomorphic. From the construction it follows that $\mathfrak{g}(D) = \Phi$.

Now we can prove the

THEOREM 12.5 (Serre [1]). *In order that Cousin's second problem should be always solvable on the holomorphically complete complex manifold Z, it is necessary and sufficient that $H^2(Z, \mathfrak{Z}) = 0$. Here $\mathfrak{Z}$ is the group of integers.*

PROOF. I. Consider the mapping θ: $\mathfrak{O}(Z) \longrightarrow \mathfrak{F}(Z)$ associating with each germ $\phi_z \in \mathfrak{O}_z$ the germ $e^{2\pi i\phi_z} \in \mathfrak{F}_z$. This homomorphism is an epimorphism (since every invertible function has a logarithm). The kernel of this homomorphism is the sheaf of holomorphic functions identically equal to integers, i.e., the constant sheaf of groups isomorphic to the group $\mathfrak{Z}$ of the integers.

In view of Theorem 0.3 the sequence

$$H^1(Z, \mathfrak{O}) \to H^1(Z, \mathfrak{F}) \to H^2(Z, \mathfrak{Z}) \to H^2(Z, \mathfrak{O})$$

is exact. By Theorem (B) (Theorem 9.9) $H^1(Z, \mathfrak{O}) = H^2(Z, \mathfrak{O}) = 0$, and therefore $H^1(Z, \mathfrak{F}) = H^2(Z, \mathfrak{Z})$. Now it is evident that by Theorem 12.3 Cousin's second problem for a holomorphically complete complex manifold Z is solvable when the condition $H^2(Z, \mathfrak{Z}) = 0$ is satisfied.

II. Necessity of this condition follows from the fact that in view of Theorem 12.4 one can find, for any element $\Phi \in H^1(Z, \mathfrak{F})$, a divisor $D \in H^0(Z, \mathfrak{D})$ such that $\mathfrak{g}(D) = \Phi$. In fact, under these conditions, from the nontriviality of the group $H^1(Z, \mathfrak{F})$ there also follows the nontriviality of the homomorphism $\mathfrak{g}$.

We note that the following proposition is true, allowing us to judge when some definite second problem of Cousin is solvable.

THEOREM 12.6. *If the cohomology group* $H^1(Z, \mathfrak{O}) = 0$ *for a complex manifold* Z, *then there exists, for the divisor* $D \in H^0(Z, \mathfrak{D})$, *a function* $f \in H^0(Z, \mathfrak{M}^2)$ *to which its germs belong at all points* $z \in Z$, *if and only if the cohomology class* $h(D) \in H^2(Z, \mathfrak{Z})$ *corresponding to this divisor is trivial.*

We shall omit the proof of this theorem (it differs only a little from the proof of Theorem 12.5).

We now consider some applications of the above criterion. Let $G = G_1 \times G_2 \times \cdots \times G_n \subset C^n$ be a polycylindrical domain, where $G_k \subset C^1_{z_k}$, $k = 1, 2, \cdots, n$, and the domains $G_2, \cdots, G_n$ are simply-connected. Then the cohomology groups of the domain G coincide with the cohomology groups of the domain G_1 since these groups do not change under the multiplication of the domain, for which they are calculated, by a simply-connected plane domain. In particular $H^2(G, \mathfrak{Z}) = H^2(G_1, \mathfrak{Z}) = 0$, since the two-dimensional cohomology group of any domain of the plane C^1 is trivial. Accordingly Cousin's second problem is solvable in the domain G.

Taking into account that the domain of holomorphy of the space C^n is always a holomorphically complete complex manifold we obtain, on the basis of Theorem 12.5, the following proposition.

THEOREM 12.7 (Oka [1], (I)). *Cousin's second problem is solvable in the domain of holomorphy of the space* C^n *homeomorphic to the polycylindrical domain* $G_1 \times \cdots \times G_n$ *if all the domains* G_k, *except possibly one, are simply-connected.*

This theorem was proved by K. Oka in 1939 without using the theory of coherent analytic sheaves.

We note that K. Stein's results [2] on the solution of Cousin's second problem may also be obtained from Theorem 12.5.

In the space C^3 of the variables z_1, z_2, z_3 consider the domain $G = \{ | z_1^2 + z_2^2 + z_3^2 - 1 | < 1 \}$. In view of Theorem 11.7 it is a domain of holomorphy. Indeed, if a point $(z_1^0, z_2^0, z_3^0) \notin G$ and $\alpha = (z_1^0)^2 + (z_2^0)^2 + (z_3^0)^2 - 1$, then $|\alpha| \geq 1$; in this case the function $f = [\alpha - (z_1^2 + z_2^2 + z_3^2 - 1)]^{-1}$ is holomorphic in the

domain G and is not continued to the point (z_1^0, z_2^0, z_3^0).

It is readily seen that the domain G is homeomorphic to the product of the unit disk by the surface $\Pi = \{z_1^2 + z_2^2 + z_3^2 = 1\}$. This surface is simply-connected (and hence the domain G is also simply-connected) and, as is proved by direct calculation, it has a nontrivial two-dimensional cohomology. Because $H^2(G, \mathfrak{Z}) = H^2(\Pi, \mathfrak{Z})$, not all Cousin's second problems are solvable in G. For example, if L is one of the connected components of the intersection $\{z_2 - iz_1 = 0\} \cap G$, then there does not exist a function holomorphic in the domain G, equal to zero on L and distinct from zero on $G \setminus L$.

Thus we have obtained the (negative) answer to the problem set up by Behnke and Thullen in 1934:[1] is Cousin's second problem in a simply-connected domain of holomorphy of the space C^n always solvable?

3. Poincaré's problem and related questions. Let f be a meromorphic function on a complex manifold Z, and let $(f) = \pi^* f \in H^0(Z, \mathfrak{D})$ be a divisor to which the germs of the function f belong at all points of the manifold Z. If f and g are two such functions, then $(fg) = (f) + (g)$ in the sense of the addition (composition) law in the group $H^0(Z, \mathfrak{D})$. By definition, each meromorphic function $f \in H^0(Z, \mathfrak{M}^2)$ may be represented in some neighborhood of any point $z \in Z$ in the form of a difference of two holomorphic functions. Therefore every divisor $D \in H^0(Z, \mathfrak{D})$ may be represented in a form $D = D_+ - D_-$. Here D_+ and D_- are positive divisors over the manifold Z (see the definition of a positive divisor in §7.2); subtraction is to be taken in the sense of the law of composition in the group $H^0(Z, \mathfrak{D})$.

THEOREM 12.8. *Any Poincaré's problem is solvable on a holomorphically complete complex manifold Z.*

PROOF. Let f be a meromorphic function given on the manifold Z and let $(f) = D_+ - D_-$, where $D_+, D_- \in H^0(Z, \mathfrak{D})$ are positive divisors. Applying Theorem 12.4 to the cohomology class $-\mathfrak{g}(D_-)$ we find that there exists a positive divisor $D' \in H^0(Z, \mathfrak{D})$ such that $\mathfrak{g}(D') = -\mathfrak{g}(D_-)$, or $\mathfrak{g}(D' + D_-) = 0$. Hence it follows that there exists a function $g \in H^0(Z, \mathfrak{M}^2)$ such that $(g) = D' + D_-$. Since the two divisors D' and D_- are positive, the function $g \in H^0(Z, \mathfrak{O})$. The function fg belongs to the divisor $(D_+ - D_-) + (D' + D_-) = D_+ + D'$. This last expression is also positive and thus the function $fg = h$ is also holomorphic on

1) See Behnke-Thullen [1], p. 68.

the manifold Z. From the equality $fg = h$ one finds that $f = h/g$ where $h, g \in H^0(Z, \mathfrak{O})$. Theorem 12.8 is proved.

We have already remarked that along with the ordinary problem of Poincaré there is the so-called *strengthened problem of Poincaré* (see §7.3). Behnke and Stein [2] investigated the possibility of solving it for domains $D \subset C^n$ and established the extent to which this possibility is connected with the solvability of Cousin's first and second problems for those domains (they considered the last problem only for positive divisors). The results of Behnke and Stein are summarized in the following table which we present without complete proof (the plus sign in this table signifies the solvability of the corresponding problem or the realizability of the corresponding hypothesis, while the minus sign means unsolvability or, correspondingly, unrealizability).

No. of hypothesis	Cousin's problem I	Cousin's problem II	Strengthened problem of Poincaré	Realizability
1	+	+	+	+
2	+	+	—	—
3	+	—	+	—
4	+	—	—	+
5	—	+	+	+
6	—	+	—	—
7	—	—	+	+
8	—	—	—	+

Hypothesis No. 1 is realized, for example, for the bicylinder $\{|z_1| < 1, |z_2| < 1\}$.

Hypotheses Nos. 2 and 6 cannot be realized, since *if Cousin's second problem for positive divisors is solvable in some domain $B \subset C^n$, then the strengthened problem of Poincaré is also solvable in it.*

Indeed, if a function $\Phi(z)$ is meromorphic in the domain $B \subset C^n$, then in some neighborhood U_P of every point $P \in B$

$$\Phi(z) = \frac{G_P(z)}{H_P(z)},$$

where the functions $G_P, H_P \in \mathfrak{O}_{U_P}$ and have no common divisor belonging to the ring $\mathfrak{O}_{U_P}$. As a result of giving the function H_P in the domain B there is defined a positive divisor $D \in H^0(B, \mathfrak{D})$; let $H(z)$ be a holomorphic function in the

domain B to which this divisor D belongs. Then the function $G(z) = \Phi(z)\, H(z)$ is holomorphic in the domain B and has no holomorphic divisor common with the function $H(z)$ everywhere in the domain B. Thus we see that $\Phi(z) = G(z)/H(z)$; we have proved the solvability of the strengthened problem of Poincaré in the domain B.

Hypothesis No. 3 cannot be realized, since *if Cousin's first problem and the strengthened problem of Poincaré are solvable in some domain $B \subset C^n$, then Cousin's second problem is also solvable in it.*

We shall not take the time here to prove this statement.

Hypothesis No. 4 is realized for the domain considered by Gronwall (see §25.3, Chapter V, (I)) and for the domain considered at the end of the preceding subsection; notice moreover that the ordinary (not strengthened) problem of Poincaré can be solved in the latter domain.

Hypothesis No. 5 is realized in the domain B_1 which is obtained from the bicylindrical domain $\{0 < |z_1| < 1,\ |z_2| < 1\}$ by adding to it the disk $\{z_1 = 0,\ |z_2| < 1/2\}$. The domain B_1 is not a domain of holomorphy, and therefore in view of Theorem 7.2 none of Cousin's first problem is solvable in it. On the other hand it can be shown (but we will not take the time to do it) that Cousin's second problem and consequently the strengthened problem of Poincaré as well are always solvable in the domain B_1.

Hypothesis No. 7 is realized in the domain B_2 which is obtained from the bicylinder $\{|z_1| < 1,\ |z_2| < 1\}$ by excluding from it points belonging to the domain

$$\left\{(|z_1| - 1)^2 + \left(|z_2| - \frac{1}{2}\right)^2 < \frac{1}{16}\right\}.$$

The domain B_2 is not a domain of holomorphy and therefore Cousin's first problem is not solvable in it. In this domain Cousin's second problem is also not solvable.

In fact, take the analytic plane $z_1 = 7/8$. Its two pieces $T_1\{z_1 = 7/8,\ |z_2| < 1/2 - \sqrt{3}/8\}$ and $T_2\{z_1 = 7/8,\ 1/2 + \sqrt{3}/8 < |z_2| < 1\}$ lie in the domain B_2. We consider the divisor $D \in H^0(B_2, \mathfrak{D})$ defined as follows:

$$D_P \ni \left(z_1 - \frac{7}{8}\right), \qquad \text{if the point } P \in T_1,$$

$$D_P \ni 1, \qquad \text{if the point } P \in B_2 \setminus T_1.$$

If there were a function $f(z)$ holomorphic in the domain B_2 such that

$(f) = D$, then this function could be analytically continued to the domain $H(B_2)$, i.e., to the entire bicylinder $\{|z_1| < 1, |z_2| < 1\}$. The function $f(z)$ would then necessarily be equal to zero (or distinct from zero) on all pieces of the analytic plane $z_1 = 7/8$ that lie in this bicylinder.

However, this does not hold in our case; accordingly Cousin's second problem as formulated above turns out to be unsolvable.

On the other hand the strengthened problem of Poincaré is always solvable in the domain B_2. In general, if this problem is solvable for some domain E, then it is solvable for all domains $E_1 \subset E$ with a meromorphy hull containing the domain E or coinciding with the domain E.

Indeed, under these conditions, any function f meromorphic in the domain E_1 may be meromorphically continued to the domain E. If this function can be represented there as a quotient of two holomorphic functions, nowhere having common holomorphic divisors, then that representation holds also in the domain E_1.

In our case the bicylinder $\{|z_1| < 1, |z_2| < 1\}$ is the meromorphy hull of the domain B_2. In it Cousin's second problem and consequently also the strengthened problem of Poincaré are solvable. Hence it follows that the last problem is solvable in the domain B_2.

Hypothesis No. 8 is realized in the domain B_3 which we obtain by excluding from the space C^2: 1) the planes $z_1 = 0$ and $z_2 = 0$; 2) the bicylindrical domain $\{1 < |z_k| < 2, k = 1, 2\}$. Cousin's first problem is not always solvable in the domain B_3 since it is not a domain of holomorphy.

An example of Cousin's second problem that is not solvable in the domain B_3 may be constructed in the same way as for the domain B_2. The strengthened problem of Poincaré is unsolvable in the domain B_3 since it is not solvable in the domain $H(B_3) = C^2 \setminus \{z_1 z_2 = 0\}$. The last result was obtained by Gronwall [1].

4. **Some properties of holomorphically complete complex manifolds.** From H. Cartan's Theorems (A) and (B) one can deduce a series of important properties of holomorphically complete complex manifolds.

THEOREM 12.9. *Let E be a submanifold properly imbedded in a holomorphically complete complex manifold Z. Then any function holomorphic on the submanifold E is the trace on this submanifold of a function holomorphic on the entire manifold Z.*

PROOF. Functions holomorphic on the manifold Z and the submanifold E constitute, respectively, the group $H^0(Z, \mathfrak{O})$ and the group $H^0(Z, \mathfrak{O}/\mathfrak{S})$. The

projection $\mathfrak{O} \to \mathfrak{O}/\mathfrak{S}$ generates a homomorphism $H^0(Z, \mathfrak{O}) \to H^0(Z, \mathfrak{O}/\mathfrak{S})$. Our assertion is correct if this homomorphism is an epimorphism; and this latter fact follows from Theorem 0.3 and the fact that $H^1(Z, \mathfrak{S}) = 0$, in view of Theorems 8.13 and 9.9 (H. Cartan's Theorem (B)).

REMARK. The assertion of the last theorem remains true in the case when E is an arbitrary analytic set imbedded normally in the manifold Z (for a normal imbedding, see §17.1, Chapter III, (I)). This is proved in a way similar to Theorem 12.9, being based however not on Theorem 8.13, but on its generalization to the case of analytic manifolds.

Theorem 12.9 holds, in particular, when the submanifold E is null-dimensional, i.e., it is a discrete set of points. In this case one must regard any function (any system of complex numbers) given on this set as holomorphic on E. As a result we obtain the

COROLLARY OF THEOREM 12.9. *Suppose that on a holomorphically complete complex manifold Z there are given a discrete set of points $\{A_i,\ i = 1, 2, \cdots\}$ and complex numbers c_i, $i = 1, 2, \cdots$. Then there exists a function $f \in H^0(Z, \mathfrak{O})$ such that $f(A_i) = c_i$ for all indices i.*

This property contains, as a special case, the property of holomorphic separability of the holomorphically complete complex manifold Z.

REMARK. From the theorem just proved it follows that the holomorphically complete complex manifold possesses the property (c′) (see §9.3). In the proof of Theorem 9.9 (H. Cartan's Theorem (B)), on which Theorem 12.9 is based, a holomorphically complete complex manifold was understood essentially to mean a complex manifold that is the union of a countable set of compacta and has the properties (a), (b) and (c). As already remarked in §9.3, the presence of the property (c) in the complex manifold follows immediately from that of the property (c′) in it.

Thus we have proved (without using H. Grauert's theorem on holomorphically complete complex manifolds) the

THEOREM 12.10. *For the complex manifolds that are countable unions of compacta and have the properties* (a) *and* (b), *the conditions* (c) *and* (c′) *are equivalent.*

Above we have formulated the corollary of Theorem 12.9. It is contained as a special case in the following theorem:

THEOREM 12.11. *On a holomorphically complete complex manifold Z we are given a discrete set of points $\{A_i, i = 1, 2, \cdots\}$. Then there exists a function $f \in H^0(Z, \mathfrak{O})$ having at each point A_i a preassigned initial segment of Taylor series (of arbitrary order r_i; the series is formed in one of the local complex coordinate systems on an element of the manifold Z containing the point A_i).*

PROOF. Let a subsheaf $\mathfrak{P}(Z)$ of the sheaf $\mathfrak{O}(Z)$ be defined in the following way: 1) $\mathfrak{P}_z = \mathfrak{O}_z$ if $z \neq A_i$ $(i = 1, 2, \cdots)$; 2) $\mathfrak{P}_{A_i} \subset \mathfrak{O}_{A_i}$ is the collection of germs of those holomorphic functions which vanish at the point A_i together with all their derivatives of orders up to and including r_i. We shall prove that the sheaf $\mathfrak{P}(Z)$ is coherent; evidently we need to carry out the proof only for the point $z = A_i$.

We choose some element U of the manifold Z on which the point A_i plays the role of an origin of coordinates and which does not contain other points of our discrete set. Let $\{f\} = \{f_1, \cdots, f_N\}$ be the collection of all monomials of degree $r_i + 1$ of the local complex coordinates on this element. All functions $\phi = \sum_{k=1}^N \alpha_k f_k$, where $\alpha_k \subset \mathfrak{O}_z$, vanish at the point $z = A_i$ along with their derivatives of orders up to and including r_i; on the other hand, every function having the indicated properties may be expressed in the form of such a sum. Therefore the $\mathfrak{O}_z$-submodule of the module $\mathfrak{O}_z$ generated at the point $z = A_i$ by the system $\{f\}$ coincides with the stalk $\mathfrak{P}_z$. At any point $z \in U$, distinct from the point A_i, one can find a nonzero function f_k from the system $\{f\}$; therefore the $\mathfrak{O}_z$-submodule of the module $\mathfrak{O}_z$ generated at such a point z by the system $\{f\}$ coincides with $\mathfrak{O}_z = \mathfrak{P}_z$. Combining these deductions we conclude that $\mathfrak{P}(Z)$ is a coherent sheaf.

We now remark that equivalently we may give at points A_i, $i = 1, 2, \cdots$, sections belonging to the group $H^0(Z, \mathfrak{O}/\mathfrak{P})$ instead of initial segments of Taylor expansions up to the order r_i inclusively. The assertion of the theorem to be proved can be formulated as follows: the natural homomorphism $H^0(Z, \mathfrak{O}) \longrightarrow H^0(Z, \mathfrak{O}/\mathfrak{P})$ is an epimorphism. However, such a deduction follows from the exactness of the sequence

$$H^0(Z, \mathfrak{O}) \rightarrow H^0(Z, \mathfrak{O}/\mathfrak{P}) \rightarrow H^1(Z, \mathfrak{P})$$

(Theorem 0.3) and the equality $H^1(Z, \mathfrak{P}) = 0$, which in its turn comes out of Theorem 9.9 (H. Cartan's Theorem (B)).

From Theorems 12.9 and 12.11 there follows the

THEOREM 12.12. *A complex manifold which is a countable union of compacta is holomorphically complete if and only if the equality $H^1(Z, \mathfrak{P}) = 0$ holds for any coherent subsheaf $\mathfrak{P}(Z)$ of the sheaf $\mathfrak{O}(Z)$.*

PROOF. Necessity of the indicated condition follows from Theorem 9.9 (H. Cartan's Theorem (B)).

On the other hand, from the fact that the equality $H^1(Z, \mathfrak{P}) = 0$ holds for any coherent subsheaf $\mathfrak{P}(Z)$, it follows that the assertions of Theorems 12.9 and 12.11 are true for the manifold Z. In view of the first of these, the manifold Z has the properties (a) and (c$'$); in view of the second it has the property (b) (see §9.3). Thus Z is a holomorphically complete complex manifold.

5. Some properties of holomorphically complete complex manifolds (extension). Theorem 0.1 of de Rham can be essentially extended for the case of the holomorphically complete complex manifolds.

For a complex manifold Z we consider the sheaf $\Omega^p(Z)$ of germs of holomorphic forms of degree p. The differentiation operation carries holomorphic forms again into holomorphic forms and therefore defines the homomorphisms $d^p: \Omega^p \to \Omega^{p+1}$ and $d^{(p)}: H^0(Z, \Omega^p) \to H^0(Z, \Omega^{p+1})$. We denote by $C^p(Z)$ and $B^{p+1}(Z)$, respectively, the kernel and the image of the latter homomorphism. Evidently $B^p(Z) \subset C^p(Z)$.

We have the

THEOREM 12.13 (Serre [1]). *If Z is a holomorphically complete complex manifold, then $C^p(Z)/B^p(Z) = H^p(Z, C)$. Here C is the additive group of complex numbers.*

PROOF. Since the analytic sheaf $\Omega^p(Z)$ is coherent, $H^q(Z, \Omega^p) = 0$ for $q \geq 1$, $p \geq 0$ in view of Theorem 9.9 (H. Cartan's Theorem (B)). On the other hand, by Poincaré's well-known theorem[1)] the sequence of sheaves

$$\Omega^{p-1}(Z) \xrightarrow{d^{p-1}} \Omega^p(Z) \xrightarrow{d^p} \Omega^{p+1}(Z) \tag{2.41}$$

is exact for any $p \geq 1$. We denote by $\mathfrak{Z}^p$ the kernel of the homomorphism d^p. Then the condition of the exactness of the sequence (2.41) is equivalent to that of the sequence

$$0 \to \mathfrak{Z}^{p-1} \to \Omega^p \to \mathfrak{Z}^p \to 0.$$

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1) We have formulated Poincaré's theorem in the terminology of the theory of sheaves. For its formulation without using this terminology (and for the proof), see, for example, P. K. Raševskiĭ, *Geometrical theory of partial differential equations*, Gostehizdat, Moscow, 1947 (Russian), p. 64 ff. We remark that Poincaré's theorem asserts, roughly speaking, that a form locally closed in a euclidean space or on a manifold is always a differential.

Hence by Theorem 0.3 it follows that the sequence

$$\begin{aligned} \ldots H^i(Z, \Omega^p) &\to H^i(Z, \mathfrak{Z}^p) \\ &\to H^{i+1}(Z, \mathfrak{Z}^{p-1}) \to H^{i+1}(Z, \Omega^p) \to \cdots \end{aligned} \qquad (2.42)$$

is also exact. In view of Theorem (B) the group $H^i(Z, \Omega^p) = 0$ for $i \geq 1$ and $p \geq 0$; therefore the groups $H^i(Z, \mathfrak{Z}^p)$ and $H^{i+1}(Z, \mathfrak{Z}^{p-1})$ are isomorphic to each other; hence it further follows that the groups $H^1(Z, \mathfrak{Z}^{p-1})$ and $H^p(Z, \mathfrak{Z}^0)$ are isomorphic.

From the exactness of the sequence (2.42), when we take terms with $i = 0$, we have

$$H^0(Z, \Omega^p) \to H^0(Z, \mathfrak{Z}^p) \to H^1(Z, \mathfrak{Z}^{p-1}) \to 0.$$

We must conclude that $H^1(Z, \mathfrak{Z}^{p-1}) = C^p(Z)/B^p(Z)$. On the other hand, the kernel of the homomorphism $d^{(0)}$, the sheaf $\mathfrak{Z}^0$, coincides with the sheaf of germs of constant functions, i.e., with the group C because $H^p(Z, \mathfrak{Z}^0) = H^0(Z, C)$. Thus we obtain

$$C^p(Z)/B^p(Z) = H^p(Z, C),$$

which was to be proved.

From this theorem a series of interesting corollaries can be obtained; we mention here only one of them. On the complex manifold Z^n the holomorphic differential form of degree $p > n$ is equal to zero. Therefore, by Theorem 12.13, $H^p(Z^n, C) = 0$ for $p > n$. Hence from Theorem 0.5 one can easily obtain (see Serre [1]) the following proposition characterizing the topological nature of holomorphically complete complex manifolds (in particular, domains of holomorphy $G \subset C^n$):

COROLLARY. *For the holomorphically complete complex manifold Z^n (n being its complex dimension) every element in the homology group $H_p(Z^n, \mathfrak{Z})$ for $p > n$ has finite order. If the group $H_p(Z^n, \mathfrak{Z})$ with $p > n$ has a finite set of generators, then the p-dimensional Betti number of this manifold is equal to zero.*

We shall use Theorem 12.13 to prove Serre's theorem [2] on the Runge domain mentioned in §3, Chapter I. We repeat the statement of it.

THEOREM 3.4. *The Betti numbers of dimension greater than or equal to n are equal to zero for a Runge domain of the first kind $G \subset C^n$ which is a domain*

of holomorphy.

PROOF. It is sufficient to prove that the Betti number of dimension n is equal to zero since our Runge domain of the first kind is a domain of holomorphy and thus represents a holomorphically complete complex manifold. Hence in view of Theorem 12.13 it also follows that

$$H^n(G,\ C) = C^n(G)/B^n(G).$$

Our assertion will be proved if we establish that $C^n(G) = B^n(G)$. We introduce in the group $C^n(G)$ a topology of uniform convergence (defined in terms of compact subsets). It suffices to prove that all elements of the subgroup $B^n(G)$ together form a 1) closed and 2) everywhere dense subset in the group $C^n(G)$.

1) The subgroup $B^n(G)$ is closed in the group $C^n(G)$ not only in the case when G is a Runge domain of the first kind, but also in the case when G is an arbitrary holomorphically complete complex manifold. In fact, in order that a form $\gamma \in C^n(G)$ should belong to the group $B^n(G)$, it is necessary and sufficient that for any topologically n-dimensional cycle $V \subset G$ the equality $\int_V \gamma = 0$ should hold. The necessity of this condition follows from Stokes' theorem (see §20.2, Chapter IV, (I)). On the basis of Stokes' theorem one can also prove that the form whose integral over any cycle is equal to zero is a differential.

From Theorems 0.1 and 12.13 one can further conclude that on a holomorphically complete complex manifold a holomorphic form, being a differential of some form, is the differential of a holomorphic form. Finally it is evident that forms $\gamma \in C^n(G)$ satisfying the condition $\int_V \gamma = 0$ constitute a closed collection in the sense of the topology in question, since the last equality is preserved under a uniform passage to a limit.

2) Since G is a Runge domain of the first kind, forms of degree n of the type $P\, dz_1 \wedge \cdots \wedge dz_n$, where P is a polynomial, are everywhere dense in the group $C^n(G)$. But we always have

$$P\, dz_1 \wedge \ldots \wedge dz_n = d(Q\, dz_2 \wedge \ldots \wedge dz_n),$$

where the polynomial Q is so chosen that

$$\frac{\partial Q}{\partial z_1} = P.$$

Consequently all such forms belong to the group $B^n(G)$; hence it follows that the subgroup $B^n(G)$ is everywhere dense in the group $C^n(G)$ in the adopted topology. Thus our theorem is proved.

Theorems in this subsection indicate the necessary condition (the topological character) which must be satisfied by domains of holomorphy and Runge domains. Using the fact that the n-dimensional Betti number is necessarily equal to zero for a Runge domain of the first kind one can construct examples of domains of holomorphy of the space C^n that are not Runge domains of the first kind. An example of such a domain, $G \subset C^2$, due to K. Oka was presented in §3, Chapter I.

The above corollary of Theorem 12.13 allows us, for example, to conclude that the complement of a bounded domain of holomorphy $G \subset C^n$ is necessarily connected (see Theorem 21.2, (I)).

However, these properties are not sufficient that the domain $G \subset C^n$ should be a domain of holomorphy or a Runge domain, and thus they do not give the complete characterization of such domains. Properties which completely characterize domains of holomorphy and other domains connected with them will be considered in the next chapter.

In conclusion we note that the cited works of H. Cartan and J.-P. Serre contain many other applications of Theorems (A) and (B). We shall not discuss them here.

CHAPTER III

DOMAINS ANALYTICALLY CONVEX IN THE SENSE OF HARTOGS

§13. PLURISUBHARMONIC FUNCTIONS

1. Definition and elementary properties. Plurisubharmonic functions play an important role in the theory of domains analytically convex in the sense of Hartogs. These functions were investigated by Oka [3], (I), Lelong [1, 2, 3] and Bremermann [1, 3], (I).

DEFINITION. *A real function $\phi(z) = \phi(z_1, \cdots, z_n)$ given in a certain domain $D \subset C_z^n$ is said to be plurisubharmonic in that domain if it possesses the following properties:*

1) *The function $e^{\phi(z)}$ is finite and upper semicontinuous at all points $z \in D$.*

Thus $-\infty$ may be a value of the function $\phi(z)$ (we here assume that $\phi(z_0) = -\infty$ if $\lim_{z \to z_0}\phi(z) = -\infty$, while $e^{-\infty} = \lim_{\phi \to -\infty} e^{\phi} = 0$). Notice that from the definition 1) there follows the upper semicontinuity of the function $\phi(z)$ in the domain D.

We consider the analytic plane $E^{(\alpha)} = \{Z_k = \alpha_k t + z_k,\ k = 1, \cdots, n\}$. Here $\alpha_1, \cdots, \alpha_n$ are constants, t is a complex parameter, and the point $z = (z_1, \cdots, z_n) \in D$; evidently the point $z \in E^{(\alpha)}$ and corresponds to the value $t = 0$.

Further, let $D_{E^{(\alpha)}} = D \cap E^{(\alpha)}$, and let $D_{E_j^{(\alpha)}}$, $j = 1, \cdots, N$, be connected components of the set $D_{E^{(\alpha)}}$, while $E_j^{(\alpha)}$ are corresponding domains on the plane C_t^1 of the complex variable t. Then

2) *For any vector $\alpha = (\alpha_1, \cdots, \alpha_n)$ and any index j the trace of the function $\phi(z)$, the function $\phi(t) = \phi(\alpha_1 t + z_1, \cdots, \alpha_n t + z_n)$, is subharmonic in the*

domain $E_j^{(\alpha)}$.

We again mention the definition of a subharmonic function which we shall often use in the following pages.

A real function $\phi(t)$ given in a certain domain E of the plane C_t^1 of the variable t is said to be subharmonic if it possesses the following properties:

1) The function $e^{\phi(t)}$ is finite and upper semicontinuous at all points $t \in E$.

2) Let $p(t)$ be a function harmonic in some domain $G \Subset E$ and continuous in the closed domain $\bar{G}$. Then $p(t) \geq \phi(t)$ at all points $t \in G$ if it is so for $t \in \partial G$.

Here again the first condition states that the value $\phi(t) = -\infty$ is not excluded from consideration. We naturally assume that at those points $t \in G$ for which $\phi(t) = -\infty$ the second requirement is realized automatically.

As is well known, if the function $\phi(t)$ belongs to the class $\mathcal{C}^2$, then the condition of its subharmonicity is expressed by the inequality

$$\Delta\varphi = \frac{1}{4}\frac{\partial^2\varphi}{\partial t\,\partial\bar{t}} \geqslant 0. \tag{3.1}$$

Here Δ is the Laplace operator.

We can write out an analogous condition for the plurisubharmonicity of the function $\phi(z) \in \mathcal{C}^2$. It will be obtained if we form the inequality (3.1) for the function $\phi(t) = \phi(\alpha_1 t + z_1, \cdots, \alpha_n t + z_n)$ at the point $t = 0$. Thus we obtain the following proposition:

THEOREM 13.1. *In order that the function $\phi(z)$ of the class $\mathcal{C}^2$ should be plurisubharmonic in a domain $D \subset C_z^n$, it is necessary and sufficient that at all points $z \in D$ and for any α_j one should have*

$$H(\varphi) = \sum_{i,j=1}^{n} \frac{\partial^2\varphi}{\partial z_i\,\partial\bar{z}_j}\alpha_i\bar{\alpha}_j \geqslant 0. \tag{3.2}$$

REMARK 1. If $H(\phi) = 0$ at all points $z \in D$, then $\phi(z)$ is a pluriharmonic function in the domain D. In this case, indeed, all the mixed derivatives $\partial^2\phi/\partial z_i\partial\bar{z}_j$ vanish since the α_j are arbitrary, and hence our assertion follows.

REMARK 2. Inequality (3.2) is also a necessary and sufficient condition for the plurisubharmonicity of the function $\phi(z)$ in the general case (not only when $\phi \in \mathcal{C}^2$). Then, however, one should (if necessary) regard the mixed derivatives appearing in this inequality as corresponding generalized functions (distributions

of L. Schwartz); see Lelong [3].

COROLLARY 1 OF THEOREM 13.1. *Every plurisubharmonic function is a subharmonic function.* For the function $\phi(z) \in \mathcal{C}^2$ this is readily observed from the condition (3.2). Indeed, putting

$$\alpha_j = \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k, \end{cases}$$

we find that $\partial^2\phi/\partial z_k \partial \bar{z}_k \geq 0$ for all k from $1 \leq k \leq n$. Hence it follows that in the case being analyzed we have

$$\Delta\varphi = \frac{1}{4} \sum_{k=1}^{n} \frac{\partial^2 \varphi}{\partial z_k \, \partial \bar{z}_k} \geqslant 0,$$

and accordingly $\phi(z)$ is a subharmonic function.

A function $\phi(z)$ is said to be *plurisubharmonic at a point* $z^0 \in C^n$ *if it is plurisubharmonic in some neighborhood of that point.* Evidently (see Theorem 13.1 and its Remark 2) the function $\phi(z)$ is plurisubharmonic in a domain $D \subset C^n$ if it is plurisubharmonic at all points of that domain.

Notice that along with plurisubharmonic functions one may consider *plurisuperharmonic* functions. A function $\phi(z)$ is said to be plurisuperharmonic if the function $-\phi(z)$ is plurisubharmonic.

For $n = 1$ the class of plurisubharmonic functions evidently coincides with that of subharmonic functions.

We now point out the analogy and the connections which exist between plurisubharmonic functions and convex functions.

As is well known, a real function $\psi(\tau)$ given on a certain interval B of the axis R^1_τ of real variable τ is said to be convex (or, as it might be expressed, "sublinear") if it has the following property:

If $l(\tau) \geq \psi(\tau)$ for $\tau \in \partial G$, where $l(\tau)$ is a linear function in some closed interval $G \Subset B$, then $l(\tau) \geq \psi(\tau)$ at all points $\tau \in G$.

If the function $\psi(\tau)$ belongs to the class $\mathcal{C}^2$, then the condition of its convexity is expressed by the inequality $d^2\psi/d\tau^2 \geq 0$. This inequality is also a necessary and sufficient condition for the convexity of the function $\psi(\tau)$ in the general case (not only when $\psi \in \mathcal{C}^2$). Then, however, one should in general understand by $d^2\psi/d\tau^2$ the corresponding generalized function.

Let $\psi(z_1, \cdots, z_n)$ be a real function given in some domain $D \subset C_z^n$. We consider the straight line $L = \{Z_k = \alpha_k \tau + z_k,\ k = 1, \cdots, n\}$. Here $\alpha_1, \cdots, \alpha_n$ are complex constants, τ is a real parameter, and the point $z = (z_1, \cdots, z_n) \in D$. We form the intersection $D_L = D \cap L$. Let D_{L_j} $(j = 1, \cdots, N)$ be connected components of the set D_L, while B_j are corresponding intervals on the axis R_τ^1 of the variable τ. Then the function $\psi(z)$ is convex (or, it might be said, "plurisublinear") in the domain D if its trace, namely, the function $\psi(\tau) = \psi(\alpha_1 \tau + z_1, \cdots, \alpha_n \tau + z_n)$ is convex in all intervals B_j.

If the function $\psi(z)$ belongs to the class $\mathcal{C}^2$, then the condition of its convexity may be expressed by the condition

$$M(\psi) = \sum_{i,j=1}^{2n} \frac{\partial^2 \psi}{\partial x_i \, \partial x_j} a_i a_j = \sum_{i,j=1}^{2n} \frac{\partial^2 \psi}{\partial z_i \, \partial z_j} \alpha_i \alpha_j \geqslant 0. \qquad (3.2_1)$$

Here $z_k = x_k + i x_{n+k}$, $\alpha_k = a_k + i a_{n+k}$, $z_{k+n} = \bar{z}_k$, $\alpha_{k+n} = \bar{\alpha}_k$ $(k = 1, \cdots, n)$, and $a_1, \cdots, a_{2n}$ are arbitrary real constants.

The condition (3.2_1) is also a necessary and sufficient condition for the convexity of the function $\psi(z)$ in the general case (not only when $\psi \in \mathcal{C}^2$). Then, however, its second derivatives should in general be understood as the corresponding generalized functions. Now we can prove the

COROLLARY 2 OF THEOREM 13.1. *The convex function $\phi(z)$ is always a plurisubharmonic function.*

Indeed, the form $H(\phi)$, considered as a function of $\arg \alpha_j$ $(j = 1, \cdots, n)$, depends only on the differences $\gamma_j = \alpha_j - \alpha_1$ $(j = 2, \cdots, n)$. On the other hand, it is readily seen that

$$M(\varphi) = H(\varphi) + A(\varphi)\, e^{2i \arg \alpha_1} + \overline{A(\varphi)}\, e^{-2i \arg \alpha_1}.$$

Here $A(\phi)$ is a certain form; just like the form $H(\phi)$, it does not depend on $\arg \alpha_1$ after the substitution $\arg \alpha_j = \gamma_j + \arg \alpha_1$ $(j = 2, \cdots, n)$. Therefore $\arg \alpha_1$ can be so chosen that $M(\phi) = H(\phi)$. The form $H(\phi)$ preserves its value. Hence it follows that the form $H(\phi) \geq 0$, for any $\alpha_1, \cdots, \alpha_n$, since $M(\phi) \geq 0$. With this our assertion is proved.

In conclusion we now note some properties of plurisubharmonic functions. These come out of the corresponding properties of subharmonic functions. [1)]

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1) See I. I. Privalov, *Subharmonic functions*, GTTI, Moscow, 1937 (Russian). For the proof of these propositions, see also Lelong [1, 2] and Bremermann [1].

THEOREM 13.2. 1) *The modulus of a holomorphic function and the logarithm of the modulus of a holomorphic function are plurisubharmonic functions.*

2) *Sums of plurisubharmonic functions and limits of uniformly convergent sequences of plurisubharmonic functions are plurisubharmonic functions.*

3) *The upper bound of any set of plurisubharmonic functions is a plurisubharmonic function if it is upper semicontinuous.*

4) *A plurisubharmonic function distinct from a constant in some domain D cannot attain its upper bound in that domain.*

5) *Under a holomorphic mapping of the domain D^* in the space C^m_ζ of the variables $\zeta_1, \cdots, \zeta_m$ onto the domain D in the space C^n_z of the variables $z_1, \cdots \cdots, z_n$, to plurisubharmonic functions in the domain D there correspond plurisubharmonic functions in the domain D^*.*

The last property for functions $\phi \in \mathcal{C}^2$ follows immediately from the condition (3.2). Under a holomorphic mapping $z_k = z_k(\zeta_1, \cdots, \zeta_m)$, $k = 1, \cdots, n$, the following equality holds for the function $\phi(z) = \phi(z(\zeta))$ at each point $\zeta \in D^*$

$$\sum_{p,q=1}^{m} \frac{\partial^2 \varphi(z(\zeta))}{\partial \zeta_p \, \partial \bar{\zeta}_q} \beta_p \bar{\beta}_q = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi(z)}{\partial z_i \, \partial \bar{z}_j} \alpha_i \bar{\alpha}_j,$$

where $\alpha_i = \sum_{p=1}^m (\partial z_i / \partial \zeta_p) \beta_p$. Hence we conclude: if $\phi(z)$ is a plurisubharmonic function in the domain D, then $\phi(z(\zeta))$ is a plurisubharmonic function in the domain D^*.

Propositions 2), 5) of Theorem 13.2 are also valid for pluriharmonic functions. A theorem analogous to Theorem 13.2 can be also stated for plurisuperharmonic functions. Proposition 4) of Theorem 13.2 also holds for convex functions.

THEOREM 13.3. *Let F be a complex k-dimensional surface consisting of ordinary points and situated in a domain $D \subset C^n$ $(1 \le k \le n)$, let $\bar{S}$ be a closed domain lying on this surface, and let $\phi(z)$ be a plurisubharmonic function in the domain D. Then*

$$\sup_{z \in \bar{S}} \varphi(z) = \sup_{z \in \partial S} \varphi(z).$$

REMARK. In other words, Theorem 13.3 asserts that the maximum principle (see §4.7, Chapter I, (I)) is valid for plurisubharmonic functions on the analytic surface.

PROOF OF THEOREM 13.3. Evidently the assertion of the theorem is true if the function involved is a constant. Suppose that this assertion is invalid for a

plurisubharmonic function $\phi(z)$ distinct from a constant. We denote by m the upper bound of values of the function $\phi(z)$ in the closed domain $\bar{S}$, and let $M = [\{\phi(z) = m\} \cap \bar{S}] \Subset S$. Take a point $z^{(0)} \in \partial M$. Since $z^{(0)}$ is an ordinary point of the surface F, we may represent the surface in the neighborhood of this point by the equations $z_p = z_p(\zeta_1, \cdots, \zeta_k)$, $p = 1, \cdots, n$, with the following properties: 1) to the point $z^{(0)}$ there correspond null values of the parameters $\zeta_1, \cdots, \zeta_k$; 2) the function $z_p(\zeta)$ is holomorphic in the neighborhood U of the origin of coordinates in the space C^k_ζ corresponding to the above-mentioned neighborhood of the point $z^{(0)}$ on the surface F. Then it is found that the plurisubharmonic function $\phi(z(\zeta))$ is distinct from a constant in the domain U and attains its largest value at an interior point of this domain. We are thus led to a contradiction to Proposition 4) of Theorem 13.2 and so must reject our supposition. Theorem 13.3 is proved.

2. **Fundamental property of domains analytically convex in the sense of Hartogs.** The role of plurisubharmonic functions in the theory of domains analytically convex in the sense of Hartogs is derived from the following theorem:

THEOREM 13.4. *Let $D \subset C^n$ be a domain analytically convex in the sense of Hartogs. Then $-\ln d_D(z)$, where $d_D(z)$ is the euclidean distance from a point $z \in D$ to the boundary ∂D, is a plurisubharmonic function.*

We shall prove this theorem for the case of a bounded domain D in the space $C^2_{w,z}$ of the variables w and z.[1] We preface the proof with a series of lemmas.

LEMMA 1. *Let D be a domain in the space $C^2_{w,z}$, analytically convex in the sense of Hartogs. Then, if $\{w = 0, |z| \le r_1\} \subset D$ and $\{|w| \le r, |z| = r_1\} \subset D$, we also have $\{|w| < r, |z| < r_1\} \subset D$.*

PROOF. We shall prove the lemma by contradiction. Suppose that a point $(\omega, \zeta) \in \partial D$, although $|\omega| < r$ and $|\zeta| < r_1$. Set $|\omega| = \rho$. Carry out the mapping

$$W = \frac{k}{w}, \quad Z = z,$$

where $k = r_1 r\rho/\sqrt{r^2 - \rho^2}$. As a result of this mapping the domain D goes over into a domain D^* of the extended space of the variables W and Z, analytically convex in the sense of Hartogs at all of its finite boundary points (see the second

1) For the complete proof of Theorem 13.4 see, for example, Bremermann [3] (I).

requirement in the definition of analytic convexity in the sense of Hartogs in §12.1, Chapter II, (I)).

Since the closed disk $\{w = 0,\ |z| \le r_1\} \subset D$, there exists a number σ, $0 < \sigma < \rho$, such that the closed bicylinder $\{|w| \le \sigma,\ |z| \le r_1\} \subset D$ as well. Hence it follows that $\{|W| \ge k/\sigma,\ |Z| \le r_1\} \subset D^*$. Since the closed set $\{|w| \le r,\ |z| = r_1\} \subset D$, we have $\{|W| \ge k/r = r_1\rho/\sqrt{r^2 - \rho^2},\ |Z| = r_1\} \subset D^*$. Finally, by assumption, the point $(k/\omega, \zeta)$ lying in the domain

$$\mathfrak{S} = \left\{ \frac{k}{\sigma} > |W| > \frac{r_1\rho}{\sqrt{r^2-\rho^2}},\quad |Z| < r_1 \right\}$$

belongs to the boundary ∂D^* of the domain D^*.

We consider the set $E = \partial D^* \cap \overline{\mathfrak{S}}$. It is not empty since the point $(k/\omega, \zeta) \in E$; it is perfect since $\overline{\mathfrak{S}}$ is a closed and bounded domain. The intersection $\partial\mathfrak{S} \cap \partial D^* \neq \emptyset$, since the intersection $\mathfrak{S} \cap \partial D^* \neq \emptyset$ (see Theorem 12.5, (I)). On the other hand, the boundary $\partial\mathfrak{S}$ consists of three parts defined by the conditions:

1) $$\left\{ \frac{k}{\sigma} \geqslant |W| \geqslant \frac{r_1\rho}{\sqrt{r^2-\rho^2}},\quad |Z| = r_1 \right\};$$

2) $$\left\{ |W| = \frac{k}{\sigma},\quad |Z| < r_1 \right\};$$

3) $$\left\{ |W| = \frac{r_1\rho}{\sqrt{r^2-\rho^2}},\quad |Z| < r_1 \right\}.$$

As has been extablished above, the first and second parts of $\partial\mathfrak{S}$ lie inside the domain D^*. Therefore the points of ∂D^* must lie on the part of the boundary $\partial\mathfrak{S}$ defined by the condition 3).

Let R be the maximal distance from the points of the set E to the origin of coordinates. The point $(k/\omega, \zeta) \in E$, and therefore

$$R \geqslant \sqrt{\frac{k^2}{\rho^2} + |\zeta|^2} \geqslant \frac{k}{\rho} = \frac{rr_1}{\sqrt{r^2-\rho^2}}.$$

Let a point $(W_0, Z_0) \in E$ be separated from the origin of coordinates by a distance R (such a point must exist because the set E is perfect). It cannot lie on $\partial\mathfrak{S}$, since otherwise we would have

$$|W_0| = \frac{r_1\rho}{\sqrt{r^2-\rho^2}},\quad |W_0|^2 + |Z_0|^2 = R^2$$

or

$$|Z_0| = \sqrt{R^2 - \frac{r_1^2\rho^2}{r^2-\rho^2}} \geqslant \sqrt{\frac{r^2r_1^2}{r^2-\rho^2} - \frac{\rho^2r_1^2}{r^2-\rho^2}} = r_1,$$

which is impossible because we must have $|Z| < r_1$ in the part of the boundary $\partial\mathfrak{E}$ defined by the condition 3). Therefore the point (W_0, Z_0) must belong to the domain $\mathfrak{E}$, which is impossible in view of Theorem 12.3. The lemma is proved.

Suppose that holomorphic functions $w' = \phi(w)$ and $z' = \psi(z)$ biholomorphically transform the closed disks $\{|w| \leq r\}$ and $\{|z| \leq r_1\}$ onto the closed bounded domains $\mathfrak{D} \subset C^1_w$ and $\mathfrak{D}_1 \subset C^1_z$. These functions together define a biholomorphic mapping of the closed bicylinder $\{|w| \leq r, |z| \leq r_1\}$ onto the closed domain $\overline{\mathfrak{D}} \times \overline{\mathfrak{D}}_1$. Using the fact that analytic convexity in the sense of Hartogs is preserved under biholomorphic mappings, we can obtain from Lemma 1 the following proposition:

LEMMA 2. *Let D be a domain of the space $C^2_{w,z}$, analytically convex in the sense of Hartogs, and let $\mathfrak{D} \subset C^1_w$, $\mathfrak{D}_1 \subset C^1_z$ be bounded domains with real analytic boundaries. If $\{w = w_0, z \in \overline{\mathfrak{D}}_1\} \subset D$ and $\{w \in \overline{\mathfrak{D}}, z \in \partial\mathfrak{D}_1\} \subset D$, where w_0 is a certain fixed point of the domain $\mathfrak{D}$, then the domain $\mathfrak{D} \times \mathfrak{D}_1 \subset D$.*

REMARK. For the application of this lemma we usually take a disk as the domain $\mathfrak{D}$.

LEMMA 3. *Let D be a bounded domain of the space $C^2_{w,z}$, analytically convex in the sense of Hartogs, let $d_{z_0}(w_0)$ be the radius of the largest disk lying in the plane $z = z_0$ with its center at a point $(w_0, z_0) \in D$ and imbedded in the domain D, and let $\mathfrak{D}_j$ $(j = 1, \cdots, N)$ be connected components of the open set obtained by projecting the intersection $D \cap \{w = w_0\}$ on the z-plane. Then $-\ln d_z(w_0)$ is a subharmonic function of z in each of the domains $\mathfrak{D}_j$.*

REMARK. If D is a domain of holomorphy (meromorphy) of a certain function $f(w, z)$, the quantity $d_z(w)$ is called the radius of holomorphy (meromorphy) of that function. In this case Lemma 3 expresses the so-called *fundamental* or *characteristic* property of the radii of holomorphy and meromorphy, which was found by Hartogs in the case of the radius of holomorphy (see §1.5, Chapter I) and by Levi in the case of the radius of meromorphy.

PROOF OF LEMMA 3. I. The function $[d_z(w_0)]^{-1}$ is finite in the domain $\mathfrak{D}_j$. To prove its upper semicontinuity, we take some point $\zeta \in \mathfrak{D}_j$ and a sequence $\zeta_n \to \zeta$ for $n \to \infty$. Let α be the limit of one of the subsequences of the sequence $d_{\zeta_n}(w_0)$ as $n \to \infty$. We denote terms of this subsequence and corresponding points ζ by the same symbols $d_{\zeta_n}(w_0)$ and ζ_n as before. Then $\lim_{n\to\infty} d_{\zeta_n}(w_0) = \alpha$. We shall show that always $d_\zeta(w_0) \leq \alpha$. Consider a sequence of points $(w_n, z_n) \in \partial D$

lying on the corresponding circles $\{|w - w_0| = d_{\zeta_n}(w_0),\ z = \zeta_n\}$, $n = 1, 2, \cdots$. The domain D is assumed to be bounded; therefore this set necessarily has limit points; let (w_0, z_0) be one of them. As a limit of a sequence of boundary points of the planar domain $\{w = w_0,\ z \in \mathfrak{D}_j\}$, this point is itself a boundary point. Its distance to the point ζ is equal to α and it does not belong to the domain $\{w = w_0,\ z \in \mathfrak{D}_j\}$; hence it follows that $d_\zeta(w_0) \leq \alpha$. Therefore the upper semicontinuity of the function under consideration is proved.

II. We shall show that the function $-\ln d_z(w_0)$ satisfies the second requirement in the definition of subharmonic functions. Let a domain $G \Subset \mathfrak{D}_j$, and let a function $p(z)$ be continuous in the closed domain $\overline{G}$ and harmonic in the domain G. First assume that

$$p(z) > -\ln d_z(w_0) \text{ for } z \in \partial G. \tag{3.3}$$

It will be found that then

$$p(z) \geq -\ln d_z(w_0) \text{ for } z \in G. \tag{3.3$_1$}$$

Let $G_k \subset C_z^1$ $(k = 1, 2, \cdots)$ be a principal sequence of domains approximating the domain G from the inside. Since the function $p(z)$ is continuous, while the function $-\ln d_z(w_0)$ is upper semicontinuous in the closed domain $\overline{G}$, inequality (3.3) remains valid at the points $z \in \partial G_k$ for sufficiently large values of the number k. Denote by $q(z)$ a harmonic function conjugate to the function $p(z)$; set

$$\phi(z) = p(z) + iq(z), \quad \psi(z) = e^{\phi(z)}.$$

These functions $\phi(z)$ and $\psi(z)$ are holomorphic in the domain $\overline{G}_k$ and $\psi(z) \neq 0$.

We consider the biholomorphic mapping of the open set $H = (C_w^1 \times G_k) \cap D$ into the space of the variables W, Z:

$$W = (w - w_0)\psi(z), \quad Z = z;$$

we denote by H^* the image of the set H under this mapping. The connected components of the set H, as intersections of domains analytically convex in the sense of Hartogs, are analytically convex in the sense of Hartogs. Hence it follows that the connected components of the set H^* are analytically convex in the sense of Hartogs.

Evidently, 1) $\{w = w_0,\ z \in \bar{G}_k\} \subset H$; 2) $\{|w - w_0|^{-1} \leq \psi(z),\ z \in \partial G_k\} \subset H$ (this follows from inequality (3.3) valid at $z \in \partial G_k$ for sufficiently large values of k). Consequently: 1) $\{W = 0,\ Z \in \bar{G}_k\} \subset H^*$; 2) $\{|W| \geq 1,\ Z \in \partial G_k\} \subset H^*$. Therefore in view of Lemma 2 we also have $\{|W| \geq 1,\ Z \in G_k\} \subset H^*$ or

$$\{|w - w_0|^{-1} \leq |\psi(z)|,\ z \in G_k\} \subset H.$$

Hence it follows that inequality (3.3_1) is realized for $z \in G_k$, and accordingly also for all points $z \in G$.

Assume that the function $p(z)$ for $z \in \partial G$ satisfies only inequality (3.3_1). We shall establish that this inequality also holds for functions of the form $p(z) + \epsilon_\nu \pi(z)$, $z \in G$, which approximate the function $p(z)$, where $\epsilon_\nu > 0$, $\lim_{\nu \to \infty} \epsilon_\nu = 0$ and $\pi(z)$ is a harmonic polynomial which is positive in the domain $\bar{G}$.

LEMMA 4. *For a bounded domain $D \subset C^2_{w,z}$ which is analytically convex in the sense of Hartogs, the quantity $-\ln d_z(w)$ is a plurisubharmonic function in D.*

PROOF. I. The function $[d_z(w)]^{-1}$ is finite in the domain D. To prove its upper semicontinuity, we take some point $(\omega, \zeta) \in D$; let $\{(\omega_n, \zeta_n),\ n = 1, 2, \cdots\}$ be any sequence of points converging to the point (ω, ζ), and let α be the limit of one of the subsequences of the sequence $\{d_{\zeta_n}(\omega_n)\}$ as $n \to \infty$. We denote terms of this subsequence by the same symbols $d_{\zeta_n}(\omega_n)$ as before; then $\lim_{n\to\infty} d_{\zeta_n}(\omega_n) = \alpha$. We shall show that always $d_\zeta(\omega) \leq \alpha$. Consider a set of points $(\omega_n, \zeta_n) \in \partial D$ lying on the corresponding circles $\{|w - \omega_n| = d_{\zeta_n}(\omega_n),\ z = \zeta_n\}$, $n = 1, 2, \cdots$. The domain D is assumed to be bounded; therefore this set necessarily has limit points; let (ω_0, ζ) be one of them. As the limit of a sequence of boundary points of the domain D, this point is itself a boundary point; its distance from the point (ω, ζ) is equal to α. Hence it follows that $d_\zeta(\omega) \leq \alpha$; therefore the upper semicontinuity of the function under consideration is proved.

II. We shall now show that the function $-\ln d_z(w)$ satisfies the second condition for the plurisubharmonicity. Take an arbitrary point of the domain D and assume for conciseness of notation that it is the origin of coordinates. We shall show that in some neighborhood of this point, in any analytic planes passing through this point, the trace of the function $-\ln d_z(w)$ is a subharmonic function.

II_1. First we consider the trace of the function $-\ln d_z(w)$ in the plane $w = 0$, i.e., the function $-\ln d_z(0)$. Its subharmonicity follows immediately from Lemma 3.

II_2. Next we consider the trace of the function $-\ln d_z(w)$ in the plane $w - \alpha z = 0$ $(\alpha \neq 0)$. Change the variables by setting $W = w - \alpha z$, $Z = z$. As a result of such a mapping the domain D goes over into a domain D^* in the space of the variables W, Z, which is again analytically convex in the sense of Hartogs. In addition

$$d_z^*(w)\,|_{w=0} = d_z(w)\,|_{w=\alpha z}$$

(for fixed z our mapping is a translation). Since by Lemma 3 the function $-\ln d_z^*(W)|_{W=0}$ is a subharmonic function of Z in the neighborhood of the origin, the trace of the function $-\ln d_z(w)$ in the plane $w = \alpha z$ is a subharmonic function in the neighborhood of the origin of coordinates.

II_3. It remains to consider the trace of the function $-\ln d_z(w)$ in the plane $z = 0$. The function $d_0(w)$ is the lower bound of the distances of the point $(w, 0)$ to the points of the set $E = (\partial D) \cap \{z = 0\}$. This set E is not empty since the domain D is bounded. Then

$$-\ln d_0(w) = \sup_{(\omega,\, 0) \in E} [-\ln |w - \omega|].$$

The functions $\ln |w - \omega|$ are harmonic for all ω. Their upper bound $-\ln d_0(w)$ is upper semicontinuous (this follows from part I of the proof and the fact that a function semicontinuous in the domain D is upper semicontinuous in all the plane sections of this domain), and accordingly it is a subharmonic function. Lemma 4 is completely proved.

COMPLETION OF PROOF OF THEOREM 13.4. To the domain D we apply the mapping

$$W = \alpha w - \beta z, \quad Z = \bar{\beta} w + \bar{\alpha} z, \tag{3.4}$$

where $|\alpha|^2 + |\beta|^2 = 1$. We denote by D^* the image of the domain D under this mapping. As is easily seen, $d_D(w, z) = d_{D^*}(W, Z)$.

We consider the quantity $-\ln d_Z^*(W)$ (it has the same meaning for the domain D^* as the quantity $-\ln d_z(w)$ has for the domain D). It is a plurisubharmonic function of the variables W, Z; evidently the function $\psi(w, z) = -\ln d_Z^*(W)$ is also plurisubharmonic (with respect to the variables w and z in the domain D).

Let points $(w_1, z_1) \in D$ and $(w_2, z_2) \in \partial D$, while (W_1, Z_1) and (W_2, Z_2) are the images of these points under the mapping (3.4). We choose the parameters α and β in such a way that the equality $\bar{\beta}(w_1 - w_2) + \bar{\alpha}(z_1 - z_2) = 0$ holds. Then $Z_1 = Z_2$ and

$$|W_2 - W_1| \geqslant d^*_{Z_1}(W_1)$$

or

$$-\frac{1}{2}\ln(|w_2 - w_1|^2 + |z_2 - z_1|^2) \leqslant -\ln d^*_{Z_1}(W_1).$$

The equality sign is achieved for points (W_2, Z_2) nearest to the point (W_1, Z_1) (here $Z_2 = Z_1$). Hence it follows that $-\ln d_D(w_1, z_1)$ or equivalently (since (w_1, z_1) is an arbitrary point of the domain D), that $-\ln d_D(w, z)$ is the upper bound of a certain set of plurisubharmonic functions. This upper bound is upper semicontinuous (see the supplement to Theorem 13.4 stated below) and therefore in view of Theorem 13.2 it is again a plurisubharmonic function. Theorem 13.4 is proved.

COROLLARY. *If a domain $D \subset C^n$ is convex in the sense of Hartogs, then there exists in this domain a plurisubharmonic function $\phi(z)$ which becomes infinitely large when the point $z \in D$ approaches the boundary of the domain D in any way* (this means that $\{\phi(z) < M\} \Subset D$ for any number M; see §5.2, Chapter I).

Indeed, the function $-\ln d_D(z)$ tends to infinity when the point z aproaches the finite boundary of the domain D in any way. Therefore the corollary is obvious if the domain D is bounded. If D is an unbounded domain, the function $-\ln d_D(z)$ may remain finite on the approach of the point z to an infinite boundary point of the domain D. In this case one must consider the plurisubharmonic function $\phi(z) = \sup[-\ln d_D(z), \ln|z_1|, \cdots, \ln|z_n|]$. It clearly has the required property.

We have proved above (see §5.1, 2 and 4, Chapter I) that the kernel function $K_D(z, \bar{z})$ of the bounded domain of holomorphy is plurisubharmonic and, in the cases described in §5, becomes infinitely large when the point z approaches the boundary of the domain D in any way. Below (see §14 of the present chapter) we shall show that every domain convex in the sense of Hartogs is a domain of holomorphy. Therefore the kernel function $K_D(z, \bar{z})$ of the domain D is another example of a plurisubharmonic function which has the required property with respect to the domain of the indicated class.

SUPPLEMENT TO THEOREM 13.4 (Bremermann [3], (I)). *If the domain D is distinct from the whole space C^n, then the function $d_D(z)$ is continuous in that domain.*

PROOF. So long as $D \neq C^n$, the function $d_D(z)$ is finite at any point $z \in D$. Consider two points $z^{(1)}, z^{(2)} \in D$ and an arbitrary point $z^{(3)} \in C^n$; we shall

denote by $|z^{(k)} - z^{(l)}|$ the euclidean distance between corresponding points.

Evidently the point $z^{(3)} \in D$ for $|z^{(3)} - z^{(1)}| < d_D(z^{(1)})$. If $|z^{(1)} - z^{(2)}| < \epsilon$, then in view of the triangle inequality we have $|z^{(3)} - z^{(1)}| < |z^{(3)} - z^{(2)}| + \epsilon$. Therefore the point $z^{(3)} \in D$ for $|z^{(3)} - z^{(2)}| < d_D(z^{(1)}) - \epsilon$. Hence we conclude that for $|z^{(1)} - z^{(2)}| < \epsilon$

$$d_D(z^{(2)}) \geqslant d_D(z^{(1)}) - \varepsilon.$$

Interchanging the points $z^{(1)}$ and $z^{(2)}$ in this argument, we find that for $|z^{(1)} - z^{(2)}| < \epsilon$

$$d_D(z^{(1)}) \geqslant d_D(z^{(2)}) - \varepsilon.$$

Combining these inequalities, we obtain the result that for $|z^{(1)} - z^{(2)}| < \epsilon$

$$|d_D(z^{(1)}) - d_D(z^{(2)})| < \varepsilon.$$

Therefore our assertion is proved.

3. **Level surfaces of plurisubharmonic functions.** First of all we shall prove Theorem 13.5, which is a converse to Theorem 13.4. It will play an essential role in the following exposition.

THEOREM 13.5. *If the function* $-\ln d_D(z)$ *is plurisubharmonic in a domain* $D \subset C^n$, *then this domain* D *is analytically convex in the sense of Hartogs.*

PROOF. I. Consider bounded domains G_μ $(\mu = 0, 1, 2, \cdots)$, each lying along with its boundary on a complex k-dimensional $(1 \leq k \leq n)$ analytic surface F_μ, and let $\lim_{\mu \to \infty} F_\mu = F_0$, $\lim_{\mu \to \infty} G_\mu = G_0$, $\lim_{\mu \to \infty} \partial G_\mu = \partial G_0$, $\partial G_0 \subset D$. Evidently, then, $\partial G_\mu \subset D$ for $\mu > M$, where M is a certain number. Assume that in the domain G_0 there are points not belonging to the domain D; *we shall show that then all the domains* G_μ *for* $\mu > M_1$, *where* M_1 *is a certain number, contain points not belonging to the domain* D.

We shall demonstrate this by contradiction. Suppose that there exists a subsequence of domains $G_{\mu_j} \subset D$, where $\mu_j \to \infty$. We replace the sequence $\{G_\mu\}$ by this subsequence and include in it only domains with numbers $\mu_j > M_1$. We denote domains belonging to this new sequence by the same symbols G_μ $(\mu = 1, 2, \cdots)$ as before. Then, for $\mu \neq 0$, the inclusion relation $\overline{G}_\mu \subset D$ holds, and in view of Theorem 13.3

$$\sup_{z \in \overline{G}_\mu} [-\ln d_D(z)] = \sup_{z \in \partial G_\mu} [-\ln d_D(z)],$$

or

$$\inf_{z\in \overline{G}_\mu} d_D(z) = \inf_{z\in \partial G_\mu} d_D(z). \tag{3.5}$$

The function $d_D(z)$ is continuous in the domain D; therefore inequality (3.5) is preserved under the passage to the limit as $\mu \longrightarrow \infty$. Thus

$$\inf_{z\in \overline{G}_0} d_D(z) = \inf_{z\in \partial G_0} d_D(z).$$

Since $\partial G_0 \subset D$, we have $\inf_{z \in \partial G_0} d_D(z) > 0$ and accordingly $\inf_{z \in \overline{G}_0} d_D(z) > 0$. Hence, taking account of the geometric meaning of the quantity $d_D(z)$ we conclude that $G_0 \subset D$. We are led to a contradiction and must reject the above supposition. Our assertion is proved.

II. Now we can prove that the domain D is convex in the sense of Hartogs (see the definition of convexity in the sense of Hartogs in §12.1, Chapter II, (I)). If a point $a(a_1, \cdots, a_n) \in \partial D$, while the deleted disk $\{0 < |z_1 - a_1| < \epsilon, z_j = a_j, j = 2, \cdots, n\} \subset D$, we take as the domains G_μ the disks

$$\{|z_1 - a_1| < \varepsilon, \quad z_j = a_j^{(\mu)}, \quad j = 2, \ldots, n\}, \quad \lim_{\mu\to\infty} a_j^{(\mu)} = a_j.$$

Using the result obtained in the first part of the proof, we conclude that for sufficiently large values of μ the disks G_μ must contain points not belonging to the domain D. The assumption that at the point a the first condition for convexity in the sense of Hartogs is not satisfied for the domain D obviously contradicts this conclusion.

Under biholomorphic mappings the disks G_μ are transformed into domains lying on complex one-dimensional analytic surfaces. In the first part of the proof we have established that these domains also have the required property. In this way the second condition for convexity in the sense of Hartogs is also satisfied at the point a. Thus, since a is an arbitrary point of the boundary of the domain D, its convexity in the sense of Hartogs is proved.

From Theorems 13.4 and 13.5 there follows the

COROLLARY 1. *A domain $D \subset C^n$ is analytically convex in the sense of Hartogs if and only if the function $-\ln d_D(z)$ is plurisubharmonic.*

We remark that convex domains also have an analogous property: a domain D of the space R_n of real variables $x_1, \cdots, x_n$ is convex if and only if the function $-\ln d_D(x_1, \cdots, x_n)$ is convex (see Bremermann [3], (I)). Therefore in view

of Corollary 2 of Theorem 13.1 we have

COROLLARY 2. *The convex domain* $D \subset C^n$ *is always analytically convex in the sense of Hartogs.*

This proposition may also be obtained from other arguments (see the remark at the end of subsection 3 of the present section).

THEOREM 13.6. *If* $\phi(z)$ *is a plurisubharmonic function in a domain* $B \subset C^n$, N *is a certain real number, and the open set* $\{\phi(z) < N\} \Subset B$, *then every domain* D *which is a connected component of the set* $\{\phi(z) < N\}$ *is analytically convex in the sense of Hartogs.*

Proof of this theorem is analogous in many respects to that of the preceding theorem.

I. We preserve the notations introduced in the proof of Theorem 13.5. Then by applying Theorem 13.3 to the function $\phi(z)$ (instead of the function $-\ln d_D(z)$), we obtain (instead of equality (3.5)) the relation

$$\sup_{z \in \overline{G}_\mu} \varphi(z) = \sup_{z \in \partial G_\mu} \varphi(z). \tag{3.6}$$

Because of the upper semicontinuity of the function $\phi(z)$, equality (3.6) is preserved under the limiting process for $\mu \longrightarrow \infty$. Thus

$$\sup_{z \in \overline{G}_0} \varphi(z) = \sup_{z \in \partial G_0} \varphi(z) < M,$$

so long as $\partial G_0 \subset D$. From the inequality $\sup_{z \in \overline{G}_0} \phi(z) < M$ and the method of defining the domain D it follows that $G_0 \subset D$; this inclusion contradicts the assumption that the domain G_0 contains points not belonging to the domain D. Consequently, in the present case there also exists a number M_1 such that the domain G_μ for $\mu > M_1$ contains points $z \notin D$.

II. The second part of the proof of Theorem 13.5 is preserved in this case without any change.

COROLLARY. *If a function* $\phi(z)$ *is plurisubharmonic in a certain neighborhood* U *of a point* $z^0 \in C^n$, *then its level surface* $\phi(z) = \phi(z^0)$ *is analytically convex in the sense of Hartogs in some neighborhood of the point* z^0.

PROOF. Take a number $r > 0$ so small that the connected component D of the open set $\{\phi(z) < \phi(z^0)\} \cap \{\sum_{j=1}^n |z_j - z_j^0| < r\}$ attached to the point z^0 satisfies the condition: $D \Subset U$. Next consider the plurisubharmonic function $\phi_1(z) = \sup\,[\phi(z) - \phi(z^0), \sum_{j=1}^n |z_j - z_j^0|^2 - r^2]$. Then by Theorem 13.6 the domain D, and

consequently also the level hypersurface $\phi(z) = \phi(z^0)$ in some neighborhood of the point z^0, is analytically convex in the sense of Hartogs.

EXAMPLE. The Weil polyhedron is defined as the connected component of the open set $\{|Z_j(z)| < 1,\ j = 1, \cdots, N,\ z \in B\} = \{\phi(z) < 0\}$, where all the functions $Z_j(z)$ are holomorphic, while the function $\phi(z) = \sup [\ln |Z_1(z)|, \cdots, \ln |Z_n(z)|]$ is plurisubharmonic in the domain B (by proposition 3) of Theorem 13.2). Hence it follows that the Weil polyhedron is a domain analytically convex in the sense of Hartogs and thus it is a domain of holomorphy (in view of Theorem 14.3 to be proved below). This result was already obtained from a different direction in §5.2, Chapter I (see, further, the subsequent example for Theorem 13.8).

THEOREM 13.7. *If $\{D_\nu\}$ is a sequence of open sets, each consisting of domains analytically convex in the sense of Hartogs, while $D_\nu \subset D_{\nu+1} \subset D \subset C^n$, $\lim_{\nu \to \infty} D_\nu = D$, then the domain D is analytically convex in the sense of Hartogs.*

PROOF. Since $D_\nu \subset D_{\nu+1} \subset D$ and $\lim_{\nu \to \infty} D_\nu = D$, we have, for every point $z \in D$ and for $\nu > \nu_z$, where ν_z is some sufficiently large number,

$$-\ln d_{D_\nu}(z) \geqslant -\ln d_{D_{\nu+1}}(z) \geqslant -\ln d_D(z).$$

By Theorem 13.4 the function $-\ln d_{D_\nu}(z)$ is plurisubharmonic. In view of the assumptions of the theorem to be proved the sequence $\{-\ln d_{D_\nu}(z)\}$ converges uniformly in every domain $D^* \subset D$ to the function $-\ln d_D(z)$. Therefore by proposition 2) of Theorem 13.2 the function $-\ln d_D(z)$ is also plurisubharmonic. Hence by Theorem 13.5 it follows that the domain D is analytically convex in the sense of Hartogs.

THEOREM 13.8. *If in a domain $D \subset C^n$ there exists a plurisubharmonic function $\phi(z)$ such that for every real number M*

$$\{\varphi(z) < M\} \Subset D,$$

then the domain D is analytically convex in the sense of Hartogs.

PROOF. To establish this proposition by contradiction, one must consider a sequence of open sets $\{\phi(z) < M_\nu\}$, where $\lim_{\nu \to \infty} M_\nu = \infty$. By Theorem 13.6 the connected components of these sets are analytically convex in the sense of Hartogs. Then, by applying Theorem 13.7, we obtain the desired result.

EXAMPLE. An analytic polyhedron of general form is defined as a connected component of the open set $E = \bigcap_{j=1}^N \Delta_j$, where $\Delta_j = \{Z_j(z) \in D_j\}$. Here $Z_j(z)$ are holomorphic functions in a certain domain $D \subset C^n$, and D_j are domains on the

plane of the complex variables Z_j (see §22.1, Chapter IV, (I) or the end of §7.3, Chapter II). Consider the function $\phi(z) = \sup[-\ln d_{\Delta_1}(z), \cdots, -\ln d_{\Delta_n}(z)]$. It is plurisubharmonic by Theorems 13.6 and 13.4 and proposition 3) of Theorem 13.2 and tends to infinity on the approach of the point $z \in E$ to ∂E. Hence by Theorem 13.8 it follows that all the analytic polyhedra defined in terms of the set E are analytically convex in the sense of Hartogs [and thus they are domains of holomorphy (see Theorem 14.3 to be proved below)].

Combining Theorem 13.8 and the corollary of Theorem 13.4 we obtain the following

COROLLARY. *A domain $D \subset C^n$ is analytically convex in the sense of Hartogs if and only if there exists in the domain D a plurisubharmonic function $\phi(z)$ that becomes infinitely large on the approach of the point $z \in D$ in any way to the boundary of the domain D (i.e., $\{\phi(z) < M\} \Subset D$ for any number M;* see §5.2, Chapter I).

REMARK. In the theory of convex domains propositions similar to Theorems 13.6, 13.7, 13.8 and to the last corollary of Theorem 13.8 (see Bremermann [3], (I)) are valid. Hence Corollary 2 of Theorem 13.5 again follows from Corollary 2 of Theorem 13.1.

4. **Approximation of plurisubharmonic functions.** The condition $H(\phi) \geq 0$ (see formula (3.2)) for the plurisubharmonic function $\phi(z)$ contains second derivatives of the function $\phi(z)$ (which are, in general, to be understood in the generalized sense of L. Schwartz). Therefore the following proposition will be useful in many cases.

THEOREM 13.9. *If a function $\phi(z) \in \mathcal{C}$ is plurisubharmonic in a domain $D \subset C^n$, then for every open set $D^* \Subset D$ and any number $\epsilon > 0$ one can find a k-times continuously differentiable plurisubharmonic $\psi(z)$ such that at all points $z \in D^*$*

$$|\phi(z) - \psi(z)| < \epsilon, \quad H(\psi) > 0. \tag{3.7}$$

Here $k > 0$ is some preassigned number; it is assumed that the quanitites $\alpha_1, \cdots, \alpha_n$ in the expression $H(\psi)$ are not all equal to zero.

PROOF. We denote by r a number smaller than the boundary distance of the set D^* in the domain D (understood in the usual sense given in §11, (I)); and we let $S(z^0, r) = \{|z_j - z_j^0| < r, \ j = 1, \cdots, n\}$. Evidently $S(z^0, r) \subset D$ if the point $z^0(z_1^0, \cdots, z_n^0) \in D^*$. Consider the functions

$$\left.\begin{aligned} \varphi_1^{(r)}(z^0) &= \frac{1}{\Omega_r}\int\limits_{S(z^0,\,r)} \varphi(z)\,d\omega, \\ &\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \varphi_k^{(r)}(z^0) &= \frac{1}{\Omega_r}\int\limits_{S(z^0,\,r)} \varphi_{k-1}^{(r)}(z),\,d\omega, \end{aligned}\right\} \tag{3.8}$$

defined on a set $D_r \supset D^*$. Here Ω_r is the volume of the polycylinder $S(z^0, r)$ and $d\omega$ is the volume element of the space C^n. It can be shown that the functions (3.8) have the following property: [1] since $\phi(z) \in \mathcal{C}$, the functions $\phi_s^{(r)}(z) \in \mathcal{C}^s$, $s = 1, \cdots, k$. Since $\phi(z)$ is plurisubharmonic in the domain D, the functions $\phi_s^{(r)}(z)$ are plurisubharmonic on the open set $D_r \supset D^*$. This follows from the method of constructing the functions $\phi_s^{(r)}(z)$ by means of integrals and from proposition 2) of Theorem 13.2. Evidently, for $r \longrightarrow 0$ and $z \in D^*$ we have

$$\phi_k^{(r)}(z) \longrightarrow \phi_{k-1}^{(r)}(z) \longrightarrow \cdots \longrightarrow \phi(z).$$

Hence it is easy to conclude that we may take as the function $\psi(z)$ satisfying the first condition of (3.7) the function $\phi_k^{(r)}(z)$ corresponding to a sufficiently small value of the quantity r.

Since the function $\phi_k^{(r)}(z)$ is plurisubharmonic on the open set D^*, we have there $H(\phi_k^{(r)}) \geq 0$. We set $\psi(z) = \phi_k^{(r)}(z) + \lambda \sum_{j=1}^n z_j \bar{z}_j$, with $\lambda > 0$. Then it is evident that $H(\psi) > 0$ if the $\alpha_1, \cdots, \alpha_n$ in the expression $H(\psi)$ are not all equal to zero. By taking the quantity λ sufficiently small we also ensure that the first condition of (3.7) is satisfied. Thus the function $\psi(z)$ satisfies all the requirements of Theorem 13.9.

REMARK 1. By using an averaging process applied to the proof of Theorem 13.9 one can find a plurisubharmonic function $\psi(z)$ belonging to the class $\mathcal{C}^\infty$ and approximating the function $\phi(z)$ on the open set $D^* \Subset D$.

REMARK 2. A theorem analogous to Theorem 13.9 also holds for convex functions.

For plurisubharmonic functions of general form (not necessarily belonging to the class $\mathcal{C}$) it turns out that the following proposition (which we shall state without proof) is valid.

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1) See F. Riesz, *Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel*, Acta Math. 54 (1930), 342–345.

THEOREM 13.9$_1$ (Bremermann [1], Lelong [1]). *If a function $\phi(z)$ is plurisubharmonic in a domain $D \subset C^n$, then on every open set $D^* \Subset D$ one can find a sequence $\{\psi_\nu(z),\ \nu = 1, 2, \cdots\}$ consisting of k-times continuously differentiable plurisubharmonic functions which satisfy the condition*

$$\psi_\nu(z) \geq \psi_{\nu+1}(z) \geq \phi(z), \quad \nu = 1, 2, \cdots,$$

such that $\lim_{\nu \to \infty} \psi_\nu(z) = \phi(z)$ at all points $z \in D^*$. Here k is a certain preassigned natural number.

In several cases the following proposition will be useful.

THEOREM 13.10. *Let a function $\phi(z)$ be plurisubharmonic and continuous in a domain $D \subset C^n$ which is analytically convex in the sense of Hartogs. Then for every domain $D^* \Subset D$ and a number $\epsilon > 0$ one can find functions $f_1(z), \cdots \cdots, f_k(z)$ holomorphic in the domain D and rational numbers $c_1 > 0, \cdots, c_k > 0$ such that for $z \in D^*$*

$$\varphi(z) - \varepsilon \leqslant \sup\,[c_1 \ln |f_1(z)|,\ \ldots,\ c_k \ln |f_k(z)|] \leqslant \varphi(z).$$

We shall not dwell on the proof of this proposition (see Bremermann [1], [3], (I)).

5. **Connection with Levi's determinant.** Let a real function $\phi(w, z)$ belong to the class $\mathcal{C}^2$ in a domain $D \subset C^2_{w,z}$. For that function we form Levi's determinant (see §12, Chapter II, (I)) at a point $P(w_0, z_0) \in D$, where $[\operatorname{grad} \phi]_P \neq 0$. We find that

$$[L(\varphi)]_P = [\varphi''_{w\bar w}\varphi'_z\varphi'_{\bar z} + \varphi''_{z\bar z}\varphi'_w\varphi'_{\bar w} - \varphi''_{w\bar z}\varphi'_{\bar w}\varphi'_z - \varphi''_{\bar w z}\varphi'_w\varphi'_{\bar z}]_P$$
$$= \left[\Delta\psi\left(w,\ z_0 - \left(\frac{\varphi'_w}{\varphi'_z}\right)_P (w - w_0)\right)\right]_{w = w_0} = H(\varphi)\Big|_{\substack{\alpha_1 = (\varphi'_z)_P \\ \alpha_2 = -(\varphi'_w)_P}} \tag{3.9}$$

Thus $[L(\phi)]_P$ is the value at the point P of the Laplace operator taken on the function ϕ in the analytic plane $(w - w_0)(\phi'_w)_P + (z - z_0)(\phi'_z)_P = 0$ (this is a unique analytic plane, passing through the point P and lying in the hyperplane tangent at the point P to the hypersurface $\phi(w, z) = \phi(P)$). Hence we conclude:

If ϕ is a plurisubharmonic function in a domain $D \subset C^2_{w,z}$, then in that domain $H(\phi) \geq 0$, and consequently $[L(\phi)]_P \geq 0$ (if $H(\phi) > 0$ in the domain D, then $[L(\phi)]_P > 0$ at all points $P \in D$).

This conclusion is completely consistent with the corollary of Theorem 13.6:

The level surface of the plurisubharmonic function $\phi(w, z)$ is convex in the sense of Hartogs, and since $\phi(w, z)$ belongs to the class $\mathcal{C}^2$, it is convex in the sense of Levi. Therefore, by Theorem 12.7, (I), $L(\phi) \geq 0$ at all of its ordinary points.

The following proposition, which is to some extent converse to the above one, will also hold:

THEOREM 13.11. *If a real function $\phi(w, z)$ belongs to the class $\mathcal{C}^2$ in a domain $D \subset C^2_{w,z}$, and if the hypersurface $\Gamma = \{\phi(w, z) = 0\} \cap D$ consists of ordinary points and $L(\phi) > 0$ on Γ, then in every domain $D^* \Subset D$ this hypersurface is the level hypersurface of some plurisubharmonic function $\Phi(w, z)$, for which $H(\Phi) > 0$ in the domain D^*.*

PROOF. Let a point $P \in \Gamma$. We place the origin of coordinates at this point and for the coordinate plane $z = 0$ we take the analytic plane $(\phi'_w)_P w + (\phi'_z)_P z = 0$. Then $(\phi'_w)_P = 0$ and (see formula (3.9))

$$[L(\varphi)]_P = [\varphi''_{w\bar{w}}\varphi'_z\varphi'_{\bar{z}}] > 0.$$

Hence we find that $[\phi''_{w\bar{w}}]_P > 0$.

We may regard the hypersurface Γ as the level surface $\Phi = 0$ of the function $\Phi = \phi + q\phi^2$ (where q is some constant quantity). We shall show that $[H(\Phi)]_P > 0$ for an appropriate choice of the constant q.

In fact, it is easy to see (by the use of the fact that $\phi(P) = 0$) that at the point P

$$H(\Phi) = H(\varphi) + 2q\varphi'_z\varphi'_{\bar{z}}\alpha_2\bar{\alpha}_2 = \varphi''_{w\bar{w}}\alpha_1\bar{\alpha}_1 + \varphi''_{w\bar{z}}\alpha_1\bar{\alpha}_2 + \varphi''_{\bar{w}z}\bar{\alpha}_1\alpha_2 + (\varphi''_{z\bar{z}} + 2q\varphi'_z\varphi'_{\bar{z}})\,\alpha_2\bar{\alpha}_2.$$

We have shown above that $[\phi''_{w\bar{w}}]_P > 0$. Therefore $H(\Phi)$ will be positive if

$$[\varphi''_{w\bar{w}}(\varphi''_{z\bar{z}} + 2q\varphi'_z\varphi'_{\bar{z}}) - \varphi''_{w\bar{z}}\varphi''_{\bar{w}z}]_P > 0.$$

This last condition is realized if we take

$$2q > \frac{\varphi''_{w\bar{w}}\varphi''_{z\bar{z}} - \varphi''_{w\bar{z}}\varphi''_{\bar{w}z}}{\varphi''_{w\bar{w}}\varphi'_z\varphi'_{\bar{z}}} = -\left[\frac{D(\varphi)}{L(\varphi)}\right]_P. \tag{3.10}$$

Here $D(\phi) = \phi''_{w\bar{w}}\phi''_{z\bar{z}} - \phi''_{w\bar{z}}\phi''_{\bar{w}z}$ is the discriminant of the form $H(\phi)$. Under a biholomorphic transformation of coordinates the two determinants, $D(\phi)$ and $L(\phi)$, are multiplied by the square of the modulus of the Jacobian; their ratio

remains unchanged.

By taking q so as to satisfy the condition (3.10) we ensure the inequality $[H(\Phi)]_P > 0$. Now choose the quantity q satisfying the condition

$$2q > -\min_{P \in \Gamma \cap D^*} \left[\frac{D(\varphi)}{L(\varphi)}\right]_P .$$

Then the inequality $H(\Phi) > 0$ will hold at all points of the hypersurface $\Gamma \cap D^*$. By continuity it will also hold on some open set E which is a neighborhood of the set $\Gamma \cap D^*$.

Therefore our assertion is proved.

SUPPLEMENT TO THEOREM 13.11. The assertion of Theorem 13.11 remains valid for the hypersurface $\{\phi(w, z) = 0\}$, on which $L(\phi) \geq 0$. In this case we also have $H(\Phi) > 0$.

From this supplementary statement it follows that the inequality $L(\phi) \geq 0$ is not only necessary but also sufficient for the hypersurface of the class $\mathcal{C}^2$ to be analytically convex in the sense of Levi (see Theorem 12.7_1, (I)).

To prove the supplement to Theorem 13.11 we approximate, in the neighborhood of the point $P(w_0, z_0)$ for which $[L(\phi)]_P = 0$, the part of the hypersurface $\{\phi(w, z) = 0\}$ by a sequence of hypersurfaces $\{\phi_\nu^{(P)}(w, z) = 0\}$. Here the functions $\phi_\nu^{(P)}(w, z)$ are selected in such a way that $L(\phi_\nu^{(P)}) > 0$ in some neighborhood of the point P. Then we apply Theorem 13.7.

We shall not go into details of this proof.

6. Equivalent forms of the definitions of plurisubharmonic functions and domains analytically convex in the sense of Hartogs. The definition of a plurisubharmonic function given at the begining of this section (we shall call it the first form of the definition of a plurisubharmonic function) may be replaced by other equivalent ones. Thus, on the basis of Theorem 13.9_1, we may obtain the

Second form of the definition of a plurisubharmonic function. A real function $\phi(z)$ given in a certain domain $D \subset C_z^n$ is said to be plurisubharmonic in that domain if there exists in every open set $D^* \Subset D$ a sequence of functions $\{\phi_\nu(z)\}$ with the following properties:

1) $\lim_{\nu \to \infty} \phi_\nu(z) = \phi(z)$;
2) $\phi_\nu(z) \geq \phi_{\nu+1}(z) \geq \phi(z)$;
3) $\phi_\nu(z) \in \mathcal{C}^\infty$;

4) $\sum_{i,j=1}^{n}(\partial^2\phi_\nu/\partial z_i\partial\bar{z}_j)\alpha_i\bar{\alpha}_j \geq 0$ for any $\alpha_1, \cdots, \alpha_n$.

We may also take as the basis of the theory various other forms of the definition of a plurisubharmonic function. In particular we may base the definition of such a function on the following property (see Lelong [1, 3]).

THEOREM 13.12. *If $\phi(z)$ is a plurisubharmonic function in a domain $D \subset C^n$ and $D \Supset B_r$, where B_r is a ball of radius r with its center at a point z_0, then the spherical mean of the function $\phi(z)$, namely, the quantity*

$$\Phi(r) = \frac{(n-1)!}{2\pi^n r^{2n-1}} \int_{\partial B_r} \varphi(z)\, d\omega,$$

where $d\omega$ is the volume element of the sphere ∂B_r, is an increasing convex function of $\ln r$.

The proof of this theorem will be omitted.

The definition of a domain analytically convex in the sense of Hartogs given in §12, Chapter II, (I) (we shall call it the first form of the definition) may be replaced by other equivalent ones.

Thus, on the basis of Corollary 1 of Theorems 13.4 and 13.5, we may call a domain $D \subset C^n$ an analytically convex domain in the sense of Hartogs if the function $-\ln d_D(z)$ is plurisubharmonic (the second form of the definition) and also, on the basis of Theorems 13.6 and 13.7, we may do so if there exists a sequence of domains $\{D_\nu\}$ with the following properties:

1) $D = \lim_{\nu\to\infty} D_\nu$;

2) $D_\nu \subset D_{\nu+1} \subset D$;

3) $D_\nu = \{\phi_\nu(z) \leq 0\}$, where $\phi_\nu(z)$ is a plurisubharmonic function in the domain D for all values of ν (the third form of the definition).

On the basis of the corollaries of Theorems 13.8 and 13.5 we may call a domain $D \subset C^n$ analytically convex in the sense of Hartogs if there exists therein a plurisubharmonic function $\phi(z)$ which becomes infinitely large on the approach of the point $z \in D$ to the boundary of the domain D in any way, i.e., $\{\phi(z) < M\} \Subset D$ for any M (the fourth form of the definition). It is possible to give other definitions for the class of domains under consideration (see Lelong [2, 3]).

The domain

$$D = \{\phi(z) < 0\} \subset C^n$$

will be said to be *strictly analytically convex in the sense of Hartogs* if the

function $\phi(z)$ is plurisubharmonic in some neighborhood of the domain D and $H(\phi) > 0$ for $z \in \bar{D}$ and $\Sigma_{k=1}^{n}|\alpha_k|^2 > 0$ (by Theorem 13.6 such a domain is of course analytically convex in the sense of Hartogs).

7. **Plurisubharmonic functions in tubular domains.** Consider a tubular domain S in the space C_z^n of the variables $z_j = x_j + iy_j$, $j = 1, \cdots, n$, and its base $S^{(\mathrm{Im})} = B$ in the space $R_n^{(\mathrm{Im})}$ of the real variables $y_1, \cdots, y_n$.

THEOREM 13.13. *If a function $\phi(z)$ depends only on $y_1, \cdots, y_n$, i.e., $\phi(z) = \phi(x_1 + iy_1, \cdots, x_n + iy_n) = \phi(y_1, \cdots, y_n) = \phi(y)$, then this function $\phi(z)$ is plurisubharmonic in the neighborhood of the domain S if and only if the function $\phi(y)$ is convex in the domain B.*

PROOF. In our case

$$\sum_{i,j=1}^{n} \frac{\partial^2 \varphi(z)}{\partial z_i \partial \bar{z}_j} \alpha_i \bar{\alpha}_j = \frac{1}{4} \sum_{i,j=1}^{n} \frac{\partial^2 \varphi(y)}{\partial y_i \partial y_j} (a_i a_j + b_i b_j),$$

where $\alpha_j = a_j + ib_j$ $(j = 1, \cdots, n)$. Hence in view of the conditions (3.2) and (3.2$_1$) there follows the equivalence between plurisubharmonicity of the function $\phi(z)$ and convexity of the function $\phi(y)$.

THEOREM 13.14. *The tubular domain $S \subset C_z^n$ is analytically convex in the sense of Hartogs if and only if its base $S^{(\mathrm{Im})} = B$ is a convex domain in the space $R_n^{(\mathrm{Im})}$.*

PROOF. It is easy to see that $d_S(z) = d_B(y)$ and consequently $-\ln d_S(z) = -\ln d_B(y)$. The domain S is analytically convex in the sense of Hartogs if and only if $-\ln d_S(z)$ is a plurisubharmonic function (see the second form of the definition of a domain analytically convex in the sense of Hartogs). The domain B is convex if and only if $-\ln d_B(y)$ is a convex function (see the remark to the corollary of Theorems 13.4 and 13.5). Hence in view of the preceding theorem our assertion follows.

REMARK. From Theorem 13.13 and Theorem 14.3 it follows for the space C^n (see the next section) that the tubular domain S is a domain of holomorphy if and only if its base $S^{(\mathrm{Im})}$ is a convex domain.

This result was stated without proof in §13, Chapter II, (I). It may also be obtained immediately without reference to Theorem 14.3.

8. **Plurisubharmonic functions on complex spaces.**

DEFINITION. A function $\phi(r)$ is said to be *plurisubharmonic* on a complex space R if it possesses the following properties:

1) the function $e^{\phi(r)}$ is finite and upper semicontinuous at all points $r \in R$;

2) if r is a holomorphic mapping of some planar open set $W \subset C^1$ in the space R, then $\phi \circ r$ is a subharmonic function on the set W.

Many properties of plurisubharmonic functions in domains of the space C^n may be extended to the plurisubharmonic functions on complex spaces. We shall not discuss them (see Grauert-Remmert [1], Bremermann [3], (I)).

§14. SOLUTION OF THE HARTOGS INVERSE PROBLEM

1. Survey of results. In Chapter II, (I) it has been established that every domain of holomorphy or meromorphy in the space C^n is analytically convex in the sense of Hartogs. This is the content of the theorems of Hartogs and Levi on the continuous distribution of boundary points of domains of holomorphy and meromorphy (alternatively, on the continuous distribution of singular and essentially singular points of holomorphic functions). Naturally there arises the so-called Hartogs inverse problem: *is the domain $D \subset C^n$ analytically convex in the sense of Hartogs always a domain of holomorphy?*

For domains of the space C^2 this problem was solved affirmatively in 1942 by the Japanese mathematician Oka [3], (I) (see Theorem 14.3). Later Oka extended his result to the case of domains in the space C^n for arbitrary $n > 2$; it was also independently extended by Bremermann [1], (I) and Norguet [1], (I) (see Theorem 14.4).

As is well known, the theorems of Hartogs and Levi on the continuous distribution of boundary points of domains of holomorphy and meromorphy remain valid for domains over the space C^n that have no interiorly-branching points. Therefore the Hartogs inverse problem can also arise for such domains. In 1953 this problem was also solved in the affirmative by Oka [5, 6], (I).

From the Levi theorem on the continuous distribution of boundary points of the domain of meromorphy it follows that every domain of meromorphy over the space C^n is analytically convex in the sense of Hartogs. Therefore it follows from Oka's result that *a domain of meromorphy over the space C^n (with no interiorly-branching points) is a domain of holomorphy.*

Thus, after investigations originated by Hartogs and continued for about fifty years, it has been established that the collections of domains (over the space C^n) defined by the following headings coincide with one another: 1) holomorphy; 2) meromorphy; 3) uniform convergence of the first kind; 4) uniform convergence

of the second kind; 5) normality of the first kind; 6) normality of the second kind; 7) normal convergence; 8) holomorphic convexity; 9) analytic convexity in the sense of Hartogs; 10) convexity in the sense of H. Cartan.

In the sequel (begining with §15) these domains will usually be called *domains of holomorphy.*

The Hartogs inverse problem for interiorly-branching domains over the space P^n is not yet completely solved, though a number of important results concerning the solution of this problem for complex spaces have been obtained (see Grauert [5], Kawai [1], Narasimhan [1]). We note (as has been shown above, see §12, Chapter II, (I)) that we may construct examples of infinite interiorly-branching domains over the space P^n that are analytically convex in the sense of Hartogs but are not domains of holomorphy. 1)

In the following we shall confine ourselves to the proof of Oka's theorem for the case of domains of the space $C^2_{w,z}$.

2. **Oka's principle uniting domains of holomorphy. Beginning of the proof.** Before the publication of Oka's paper in 1942 we had no theorem about domains of holomorphy that would enable us to construct new and more extensive domains of holomorphy from given ones. In particular, the absence of such a theorem made it difficult to pass from the local properties of domains at hand (analytic convexity of a domain in the sense of Hartogs is a local property of the boundary of the domain) to the properties characterizing these domains in the large. Hence it is clear what importance is to be attached to solving the Hartogs inverse problem.

THEOREM 14.1 (Oka's principle uniting domains of holomorphy). *Let D be a bounded domain of the space $C^2_{w,z}$ containing the origin of coordinates. Let the open set D_1 be the part of the domain D lying above the hyperplane $v = a_1$, while the open set D_2 is the part of the domain D lying below the hyperplane $v = a_2$ (here $w = u + iv$ and $a_1 < 0 < a_2$). If all connected components of the open sets D_1 and D_2 are domains of holomorphy, then D is a domain of holomorphy.*

We preface the proof of this theorem with some lemmas. First we shall show that under the above conditions we can construct a domain Δ which is as close to D as desired and has several definite properties (Lemmas 1, 2, 3 and 4). Next we shall show that the domain Δ is a domain of holomorphy (Lemma 5). Hence,

1) See the introduction of the paper by Docquier-Grauert [1], (I).

on the basis of the fact that the domain D may be approximated by domains of holomorphy (see §7, Chapter I), it will follow that D also is a domain of holomorphy.

LEMMA 1. *Let $\mathfrak{S}$ be a bounded domain of holomorphy; let $\chi(w, z)$ be a certain function holomorphic in that domain and let r be a positive number. Let $\mathfrak{S}(\chi, r)$ be the set of those points P of the domain $\mathfrak{S}$ for which the boundary distance*[1] *in $\mathfrak{S}$ is greater than $r|\chi(P)|$. Then:*

1) *The set $\mathfrak{S}(\chi, r)$ consists of domains of holomorphy.*

2) *If the function $[\chi(w, z)]^{-1}$ is holomorphic and bounded in the domain $\mathfrak{S}$, then*

2a) *the set $\mathfrak{S}(\chi, r)$ is strictly contained in the domain $\mathfrak{S}$, and*

2b) *one can find an open set, containing $\mathfrak{S}(\chi, r)$ strictly in its interior, as close to $\mathfrak{S}(\chi, r)$ as desired, and consisting of domains convex with respect to the functions which are holomorphic in the domain $\mathfrak{S}$.*

PROOF. 1) $\mathfrak{S}$ is a domain of holomorphy. Let $\Phi(w, z)$ be a function having $\mathfrak{S}$ as its domain of holomorphy. We consider the family of functions

$$\Phi(w + \rho\chi(w, z)e^{i\theta},\ z + \rho'\chi(w, z)e^{i\theta'}). \tag{3.11}$$

Here ρ, ρ', θ' and θ are real parameters; $|\rho|, |\rho'| \leq r$. Let a point $P \in \mathfrak{S}(\chi, r)$. We construct the bicylinder of radius $r|\chi(P)|^{-1}$ with center at the point P. Then the values assumed by the functions of the family (3.11) at the point P coincide with those of the function $\Phi(w, z)$ in this bicylinder. Hence it follows that the set $\mathfrak{S}(\chi, r)$ is the intersection of the domains of holomorphy for the functions of the family (3.11), since it unites the points at which these functions are holomorphic. Consequently the set $\mathfrak{S}(\chi, r)$ consists of domains of holomorphy.

2a) Let the function $[\chi(w, z)]^{-1}$ be holomorphic and bounded in the domain $\mathfrak{S}$ and let $|\chi(w, z)|^{-1} < M$ or, alternatively, $|\chi(w, z)| > M^{-1}$. For the point P with boundary distance $\rho > r|\chi(P)|$, we have $\rho > rM^{-1}$. Therefore $\mathfrak{S}(\chi, r) \subset \mathfrak{S}(1, rM^{-1})$. Evidently all the open sets $\mathfrak{S}(1, \rho)$ are strictly contained in $\mathfrak{S}$. Therefore the inequality just obtained proves our assertion.

2b) Let $\{F\}$ be a collection of open sets which includes the set $\mathfrak{S}(\chi, r)$ and is convex with respect to the functions holomorphic in the domain $\mathfrak{S}$. We shall show that this collection is not empty. In fact, consider the open sets $\mathfrak{S}(1, \rho)$ for $\rho > rM^{-1}$. Each of these contains the set $\mathfrak{S}(\chi, r)$ strictly in its

1) The boundary distances in this section are to be taken as in § 11, Chapter II, (I).

interior. On the other hand all these sets are strictly contained in the domain $\mathfrak{S}$ and consist of domains of holomorphy (see Theorem 13.5, (I)), and they tend to the domain $\mathfrak{S}$ as $\rho \longrightarrow 0$. We select from these sets a sequence of domains $\mathfrak{S}(1, \rho_n)$ where $\rho_n \longrightarrow 0$. Reasoning in the same way as for the proof of the theorem on the limit of a sequence of domains of holomorphy (see Lemmas 2 and 3 in §6, Chapter I), we form a domain E, convex with respect to the functions holomorphic in the domain $\mathfrak{S}$, such that $\mathfrak{S}(1, \rho_k) \subset E \subset \mathfrak{S}(1, \rho_{k+1})$. Here k is a natural number. Evidently $E \in \{F\}$.

The intersection of the sets $\overline{F}$ again belongs to the collection $\{\overline{F}\}$. Let $\overline{F}_0$ be the intersection of all the sets $\overline{F}$. Our assertion will be proved if we show that $\overline{F}_0 = \overline{\mathfrak{S}(\chi, r)}$.

Suppose that this is not so. Consider the set $\mathfrak{S}(\chi, \rho) \supset F_0$; let r' be the largest of the numbers ρ for which $\mathfrak{S}(\chi, \rho)$ has this property. Evidently $\mathfrak{S}(\chi, r') \Supset \mathfrak{S}(\chi, (r + r')/2) \Supset \mathfrak{S}(\chi, r)$. Then there must exist an open set E_1, convex with respect to the functions holomorphic in the domain $\mathfrak{S}$, such that $\mathfrak{S}(\chi, r) \Subset E_1 \Subset \mathfrak{S}(\chi, (r + r')/2)$. The existence of the set E_1 contradicts the definitions of r' and F_0. Thus the above supposition turns out to be invalid. The proof of our lemma is complete.

In what follows we shall denote the set $\mathfrak{S}(1, \rho)$ by the symbol $\mathfrak{S}^{(\rho)}$. We denote by L, L_1, L_2 the pieces of the hyperplanes $v = 0$, $v = a_1$, $v = a_2$ that are contained in the domain D, and by D_3 the part of the domain D contained between the hyperplanes L_1 and L_2; under our assumptions about D_1 and D_2, (the definition of D_1 and D_2 is given above) the open set D_3 consists of domains of holomorphy (as an intersection of domains of holomorphy).

LEMMA 2. *One can find ν functions $Z_j(w, z)$, $j = 1, \cdots, \nu$, ν being a natural number, which are holomorphic on the sets D_3, L_1 and L_2 and have the following properties:*

1) a) *The open set $\{|Z_j(w, z)| < 1,\ j = 1, \cdots, \nu\}$ will have no points in some neighborhood of the intersection of the boundary of the domain D with the hyperplane L;*

b) *for sufficiently small $\epsilon > 0$ those points of the domain D for which $|Z_j(w, z)| < 1 - \epsilon$ $(j = 1, \cdots, \nu)$ do not all lie in some (sufficiently close) neighborhoods of the pieces of the hyperplanes L_1 and L_2;*

c) *one of the connected components of the open set, consisting of those points of the domain D which do not belong to the set D_3 and of those points of*

the set D_3 *for which all* $|Z_j(w, z)| < 1$, $j = 1, \cdots, \nu$, *extends from* L_1 *to* L_2 *(including their pieces) and contains the origin of coordinates.*

2) We denote this connected component (i.e., domain) by Δ, the set of those points of the hyperplane L for which $|Z_j(w, z)| = 1$ by Σ_j, and the boundary of the set of points of L belonging to Δ by σ, and we set $\sigma_j = \sigma \cap \Sigma_j$; then:

a) $\partial Z_j/\partial z \not\equiv 0$ *for all* $j = 1, \cdots, \nu$ *(since these manifolds* Σ_j *and* σ_j *have dimensions no larger than two)*;

b) *the intersections* $\Sigma_j \cap \Sigma_k$ *and* $\sigma_j \cap \sigma_k$ $(j \neq k)$ *have dimensions no larger than unity*;

c) $\partial Z_j/\partial z$ *does not vanish on* σ_j *with the possible exception of a finite number of points.*

3) *For any arbitrary domain* $D_0 \Subset D$, *the functions* $Z_j(w, z)$ *may be so chosen that* $D_0 \Subset \Delta \subset D$ *(in other words, the domain may be approximated from inside with any degree of accuracy by the domain* Δ *having the properties indicated above).*

REMARK 1. From Hefer's Theorem 7.3 it follows that: If points (w, z), $(w_0, z_0) \in D_3$, then

$$Z_j - Z_j^0 = (w - w_0) P_j + (z - z_0) Q_j, \quad j = 1, 2, \cdots, \nu. \tag{3.12}$$

Here $Z_j = Z_j(w, z)$, $Z_j^0 = Z_j(w_0, z_0)$ and the functions $P_j = P_j(w_0, z_0, w, z)$, $Q_j = Q_j(w_0, z_0, w, z)$ are holomorphic for (w_0, z_0), $(w, z) \in D_3$.

REMARK 2. If the second assertion of the lemma is realized the domain Δ is a Weil domain and in it one can use the corresponding integral representation (see Theorem 22.2, (I)).

PROOF. In the domain D we construct the open sets $A = D(\chi_1, \rho)$, $B = D(\chi_2, \rho)$, $C = D^{(\rho)}$, where $\chi_1 = e^{-iw}$, $\chi_2 = e^{iw}$, while ρ is a positive number chosen sufficiently small in conformity with the requirement stated below. We denote by (a), (b) and (c) the sets of those points on the hyperplane $v = v'$ ($=$ const) which belong to A, B and C, respectively. Evidently

1) if $v' = 0$, then $|\chi_1| = |\chi_2| = 1$ and $(a) = (b) = (c)$;

2) if $v' > 0$, then $|\chi_1| > 1$, $|\chi_2| < 1$ and $(a) \Subset (c) \Subset (b)$ (in more detail, if (a) is not empty, then (c) is also not empty and strictly contains (a); if (c) is not empty, then (b) is also not empty and strictly contains (c));

3) if $v' < 0$, then $|\chi_1| < 1$, $|\chi_2| > 1$ and $(a) \Supset (c) \Supset (b)$.

Let P and Q be two points of D, where the former is situated above the hyperplane L_1, while the latter is above the hyperplane L_2. We take the number ρ so small that the points P, Q and the origin of coordinates are contained in one of the connected components of the common part of the sets A and B. We denote this connected component by G. Further, let real numbers b_1 and b_2 be taken in such a way that $a_1 < b_1 < 0 < b_2 < a_2$; let L_1' and L_2' denote the pieces of the hyperplanes $v = b_1$ and $v = b_2$ lying in D. Moreover, let G_1 be the part of the domain G lying above L_1', let G_2 be the part of the domain G lying below L_2' and let G_3 be the part of the domain G situated between L_1' and L_2'. *We shall show that the open sets G_1 and G_2 consist of domains of holomorphy.*

Let A_1 be the part of A lying above L_1' and B_2 be the part of B lying below L_2'. Because of the assumptions about the sets D_1 and D_2, the sets A_1 and B_2 (ρ is supposed sufficiently small) will consist of domains of holomorphy. 1) The open set G_1 is contained in A_1. Consider the boundary of G_1. It consists of two sets of points belonging to the boundaries of A_1 and B_2, respectively. Let M be a point on the second part of the boundary of G_1. The point M is situated either on L or under L (as a result of the above-mentioned relations between (a), (b) and (c) for different v').

Let F be a closed set consisting of points of G_1 and let L_2'' be the piece of the hyperplane $v = b_2'$ (where $0 < b_2' < b_2$) lying in D. Then (since D_2 is an open set consisting of domains of holomorphy) there exists a function $f(w, z)$ holomorphic on the set D_2 such that $f(M) = 1$, while $|f(w, z)| < 1$ at 1) points F situated under L_2'' and 2) points A_1 lying in some neighborhood of L_2''. This follows from the property of holomorphic convexity of the connected components of the open set D_2; the points indicated in 1) and 2) form a subdomain of D_2 which does not contain the point M.

We further construct in A_1 a meromorphic function $\Phi(w, z)$ which is equivalent with respect to subtraction to the function $[f(w, z) - 1]^{-1}$ at points lying under L_2'' and holomorphic in the remainder of A_1. Since A_1 consists of domains of holomorphy, the construction of such a function is possible in view of Theorem 7.1. This function is holomorphic in F and has the point M as a pole.

1) The set A_1 may be considered (for sufficiently small ρ) as the common part of the open set $D_1(\chi_1, \rho)$, consisting of domains of holomorphy, and the halfspace $\operatorname{Im} w > b_1$ (which is a domain of holomorphy).

Reasoning in the same way we construct functions analogous to $\Phi(w, z)$ and having as poles points of the countable and everywhere dense covering of the part of boundary of G_1 which also belongs to the boundary of B_2. By means of these functions we construct (operating in the same way as for the proof of Cartan-Thullen's fundamental Theorem 11.6, (I)) a function $\Phi_1(w, z)$ holomorphic on the set F and having the above-mentioned part of the boundary of G_1 as its natural boundary. Multiplying $\Phi_1(w, z)$ by a function $\alpha_1(w, z)$ which has A_1 as its domain of holomorphy we obtain the function $\Phi_2 = \alpha_1\Phi_1$; its domain of holomorphy is contained in G_1, but contains the set F (if A_1 and G_1 consist of several connected components, then instead of the function $\Phi_2(w, z)$ we obtain several such functions). Hence it follows that the set F is contained in some open set consisting of domains of holomorphy and contained in G_1. Therefore G_1 also (as a limit of sequence of open sets consisting of domains of holomorphy) consists of domains of holomorphy. An analogous conclusion may also be drawn for the open set G_2.

We take a number $\delta > \rho$ sufficiently close to ρ. Consider the open sets $D' = D_3^{(\rho)}$ and $D'' = D_3^{(\delta)}$. Evidently $D'' \Subset D'$. Since $C = D^{(\rho)}$, the sets C and D' coincide between the hyperplanes L_1' and L_2' as well as in the neighborhood of these portions of the hyperplanes (the number ρ is assumed sufficiently small). Since the set D_2 consists of domains of holomorphy, to each point M of the boundary of D' there corresponds, in view of the condition of holomorphic convexity for the connected components of the boundary, a function $f(w, z)$ holomorphic on the set D_2 and a bicylinder γ_M with its center at the point M, such that $|f(w, z)| > 1$ for $(w, z) \in \gamma_M$, while $\sup|f(D')| < 1$ (see the similar argument in the proof of Theorem 2.1). In view of Borel's lemma on covering, the boundary of the set D' may be covered by a finite set of such bicylinders γ_j $(j = 1, \cdots, \nu)$; let $Z_j(w, z)$ be the functions $f(w, z)$ corresponding to these γ_j. The set $G_3 \Subset D_3$; therefore the functions $Z_j(w, z)$ are holomorphic in the neighborhood of the set G_3. We now consider the first hypothesis of our lemma.

a) Because of the smallness of the number ρ near the hyperplane L, the domain G differs little from the corresponding connected components of the set C. As was mentioned above, at points of the boundary of $C = D^{(\rho)}$ all $|Z_j(w, z)| > 1$. Thus it is established that the set $\{|Z_j(w, z)| \leq 1, \ j = 1, \cdots, \nu\}$ has no points in the neighborhood of the boundary of the domain G on the hyperplane L.

b) The part of the domain G lying on L_2' is strictly contained in the set (c)

corresponding to this hyperplane; accordingly it is strictly contained in the open set D'. If we take the number δ sufficiently close to the number ρ, then this part of the domain G is inside D''. In the neighborhood of its points all $|Z_j(w, z)| \leq 1 - \epsilon$, where ϵ is a sufficiently small number. An analogous conclusion may also be drawn for the part of the domain G lying on the portion of hyperplane L'_1.

c) Taking away from the domain G the points of G_3 not satisfying the conditions $|Z_j(w, z)| < 1$ (for all $j = 1, \cdots, \nu$) we obtain an open set. This set tends to the domain G when δ approaches ρ. We choose the number δ so close to the number ρ that this set contains the points P, Q and the origin of coordinates in one of its connected components. We denote this component by the symbol Δ.

Thus we have constructed the domain Δ arbitrarily close to the domain D and satisfying the first assertion of our lemma. We notice that here the pieces of the hyperplanes L_1 and L_2 intercepting the open sets D_1 and D_2 have turned out to be replaced by pieces of the hyperplanes L'_1 and L'_2. This is not essential for the subsequent argument, since the halfspaces $\operatorname{Im} w < b_2$ and $\operatorname{Im} w > b_1$ are domains of holomorphy and intercept from D_2 and D_1, respectively, open sets consisting of domains of holomorphy.

Lemma 2 will be proved if we establish that the domain Δ may be modified in such a way that, while continuing to have the first and third properties of this lemma, it will also satisfy the second requirement.

We now consider the second assertion of our lemma.

a) The arbitrariness in the choice of the functions $Z_j(w, z)$ evidently enables us to choose them in such a way that their derivatives with respect to z do not vanish identically.

b) Consider the manifolds $\Sigma_j = \{v = 0, |Z_j(w, z)| = 1\}$. In view of the first property their (topological) dimension does not exceed two. We shall show that for an appropriate choice of the functions $Z_j(w, z)$ the intersections $\Sigma_j \cap \Sigma_k$ cannot be more than one-dimensional. We take the numbers $r, r' > 1/2$ and consider in G_3 the manifolds

$$T_{jk} = \{v = 0, |Z_j(w, z)| = r, |Z_k(w, z)| = r'\}.$$

Each of the manifolds $\{|Z_j(w, z)| = r, v = 0\}$ within the limits of G_3 has not more than two dimensions; therefore T_{jk} has not more than one dimension with the possible exception of some definite pairs of r and r'. Evidently, in order to realize our requirement, it is sufficient, when necessary, to replace the functions

$Z_j(w, z)$ by the functions $W_j(w, z) = \alpha_j Z_j(w, z)$ respectively. Here the constants α_j are so chosen that they are all sufficiently close to unity; this allows us to preserve the properties of the domain Δ that were previously assured.

c) Now we ensure that $\partial Z_j/\partial z$ is not equal to zero at points of Σ_j (with the possible exception of a finite set of these points). Here j is any number from $1, \cdots, \nu$.

The analytic set $\partial Z_j/\partial z = 0$ may contain an analytic plane $w = \omega$ (ω being a real number). Such a plane will lie entirely in the hyperplane L. [1] To avoid this, we take the function $Z_j(w + \beta, z)$ in place of the function $Z_j(w, z)$. Here β is a purely imaginary number sufficiently small in modulus. By such a replacement we, of course, do not deprive the domain Δ of its above properties with the possible exception of the last property (the item c) of the second assertion). However, it is evident that the replacement of Z_j by $\alpha_j Z_j$ cannot affect the question whether the quantity $\partial Z_j/\partial z$ is equal to zero or not. Therefore, if necessary, we may interchange the order of the transformations b) and c) of the functions $Z_j(w, z)$. After this the (topological) dimension of the manifold $\{v=0, \partial Z_j/\partial z=0\}$ will be not more than unity. Its intersection with the manifold $|Z_j| = r$ (where $r > 1/2$) may consist of only a finite set of points (with the possible exception of some values of r). By replacing, if necessary, Z_j by $\alpha_j Z_j$ (α_j are sufficiently close to unity) we ensure the realization on Σ_j of the formulated requirement.

Thus we have constructed a domain Δ satisfying all the requirements of our lemma.

In the sequel we denote by Δ_1, Δ_2 and Δ_3 the parts of Δ situated, respectively, above L_1, below L_2 and between L_1 and L_2. [2] Evidently the open set Δ_3 consists of domains of holomorphy. In fact, the boundaries of these domains consist of the hypersurfaces $|Z_j(w, z)| = 1$ and the pieces of the hyperplanes $v = a_1$ and $v = a_2$. If M is a boundary point of Δ_3 lying on the hypersurface $|Z_j(w, z)| = 1$, then the function $[Z_j(w, z) - Z_j(M)]^{-1}$ is holomorphic in Δ_3 and

1) Generally speaking, this analytic set is decomposed into a certain collection of analytic surfaces. The analytic plane $w = \omega$ is the only analytic surface contained in the hyperplane $v = 0$.

2) We return to the previous notations (i.e., we shall write L_1 and L_2, but not L_1' and L_2').

has a pole at the point M (since $|Z_j(w, z)| < 1$ at the points of Δ_3, while $|Z_j(M)| = 1$). The halfspaces $v < a_2$ and $v > a_1$ are domains of holomorphy; there will be no difficulty in constructing functions which are holomorphic in these half-spaces (and accordingly also in Δ_3) and have the points of the pieces of the hyperplanes L_1 and L_2 as singularities. Thus in view of the corollary of Theorem 11.7, (I) the open set Δ_3 indeed consists of domains of holomorphy.

3. Continuation of the proof of Oka's principle.

LEMMA 3. *Let $\phi(w, z)$ be a certain function holomorphic in a neighborhood of S; let the functions $\psi_j(w, z, w_0, z_0)$ be defined by the equations*

$$\psi_j(w, z, w_0, z_0) = \frac{Q_j}{(w - w_0)(Z_j - Z_j^0)}, \qquad j = 1, \ldots, \nu. \tag{3.13}$$

Here Q_j are the quantities which appeared in equation (3.12). *Then the integral*

$$I(w_0, z_0) = -\frac{1}{4\pi^2} \sum_{j=1}^{\nu} \int_{\sigma_j} \psi_j(w, z, w_0, z_0)\psi(w, z)\, dw \wedge dz, \tag{3.14}$$

where the integration is taken over the outer side of the boundary of S, defines holomorphic functions in the parts of Δ_3 lying above and below the hyperplane L; these differ from each other at the points of ∂S by the function $\phi(w, z)$.

Here $S = \Delta \cap L$ and the boundary $\partial S = \bigcup_{j=1}^{\nu} \sigma_j$.

REMARK ON THE DEFINITION OF INTEGRAL (3.14). The integral (3.14) is defined in the following way: on the manifold σ_j (j is arbitrary) we take a point P such that $\partial Z_j/\partial z \neq 0$ at that point (since $\partial Z_j/\partial z$ may vanish only on a finite set of points, such a choice is always possible; in other respects the choice of the point P is arbitrary). The part of the manifold σ_j lying in the neighborhood of the point P, which we denote by $[\sigma_j]$, is represented by means of the equations

$$w = w(\alpha, \beta); \qquad z = z(\alpha, \beta). \tag{3.15}$$

Here α and β are real parameters; the right members of (3.15) are assumed to be power series in α and β (in particular, one may take u and θ_j as these parameters, where θ_j is the argument of Z_j). We shall regard α and β as the coordinates of points of a plane. Assume that equation (3.15) sets up a homeomorphic correspondence between $[\sigma_j]$ and some domain of this (α, β)-plane. Then, if some

part of the boundary of σ_j is included in the given neighborhood of P, there will correspond to it in the (α, β)-plane a finite number of analytic arcs. The order of arrangement of the parameters α and β is so chosen that $\partial(w, z)/\partial(\alpha, \beta) > 0$. Then (see §1.5, Chapter I, (I))

$$\int_{[\sigma_j]} \psi_j(w, z, w_0, z_0)\varphi(w, z)\,dw \wedge dz = \int_{[\sigma_j]} \psi_j \varphi \frac{\partial(w, z)}{\partial(\alpha, \beta)}\,d\alpha\,d\beta. \tag{3.16}$$

The second integral is extended over the domain of the (α, β)-plane corresponding to $[\sigma_j]$ under the mapping (3.15). Integral (3.14) is to be understood as the sum of corresponding integrals (3.16).

The points P for which $\partial Z_j/\partial z = 0$ should be isolated from σ by hyperspheres of sufficiently small radius ρ. Let (σ) be the part of σ_j situated in the interior of such a hypersphere. We easily find that $\int_{(\sigma)} d\sigma \to 0$ as $\rho \to 0$. Here $d\sigma$ is the surface element of (σ). Integral (3.14) is initially to be taken over the part of σ remaining after the exclusion of (σ), and then taken to the limit as $\rho \to 0$. Thus integral (3.14) in its ultimate form should, if necessary, be regarded as improper.

The function $\psi_j(w, z, w_0, z_0)$ is holomorphic in $(w, z) \in \sigma_j$ for $(w_0, z_0) \in \Delta_3\backslash S$. Indeed, since the point $(w, z) \in \sigma_j$, we have $|Z_j(w, z)| = 1$, while $|Z_j(w_0, z_0)| < 1$ since the point $(w_0, z_0) \in \Delta_3$. But if the point $(w_0, z_0) \in S$, the factor $w - w_0$ on the denominator of ψ_j will vanish at definite places of the domain of integration; the function ψ_j will not be holomorphic there. By what has been said the function $I(w_0, z_0)$ is holomorphic on the open set $\Delta_3\backslash S$. This follows from the fact that by the definition of integral (3.14) we must regard $I(w_0, z_0)$ (on the open set $\Delta_3\backslash S$) as a limit of uniformly convergent sequence of holomorphic functions.

PROOF. We shall study the behavior of the function $I(w_0, z_0)$ in the neighborhood of S. Let a point $(\omega, \zeta) \in S$ (here ω is a real number). In the w-plane we take a small disk γ with its center at the point ω; in the hyperplane L there lies only a diameter of this disk (it will be situated on the u-axis). We denote the parts of σ and σ_j projected on this diameter by σ' and σ_j', respectively, and consider the integral

$$I_1(w_0, z_0) = -\frac{1}{4\pi^2} \sum_{j=1}^{\nu} \int_{\sigma'_j} \psi_j(w, z, w_0, z_0) \varphi(w, z) \, dw \wedge dz.$$

The difference $I - I_1$ is clearly holomorphic on the set $\{w_0 \in \gamma, (w_0, z_0) \in \Delta_3\}$. Therefore it suffices to investigate the integral I_1 instead of the integral I. Set

$$\Phi_j(w, z, w_0, z_0) = \frac{Q_j}{Z_j - Z_j^0} \varphi(w, z).$$

Denote by Γ the projection on the z-plane of the section of σ by the plane $u = \omega$. Since the point $(w, z) \in \sigma_j$, we have $|Z_j| = 1$ and the function $Q_j/(Z_j - Z_j^0)$ will be holomorphic with respect to w_0, z_0 in the neighborhood of the interior points of S. Hence it follows that the function $\Phi_j(w, z, w_0, z_0)$ is holomorphic on the set $\{w = \omega, z \in \Gamma, w_0 = \omega, z_0 = \zeta\}$. Consider on the z-plane a disk γ' with its center at the point $z = \eta$ and an open set E containing Γ. We take the disks γ and γ' so small and the open set E so close to Γ that the function $\Phi_j(w, z, w_0, z_0)$ will be holomorphic on the open set $\{w \in \gamma, z \in E, w_0 \in \gamma, z_0 \in \gamma'\}$. We still note that here γ' is contained in the interior of the set $\{w \in \gamma, z \in E\}$. We now set

$$\Psi_j(z, w_0, z_0) = \Phi_j(w_0, z, w_0, z_0). \tag{3.17}$$

This function is holomorphic on the set $\{z \in E, w_0 \in \gamma, z_0 \in \gamma'\}$. In addition

$$\Phi_j(w, z, w_0, z_0) - \Psi_j(z, w_0, z_0) = (w - w_0) \chi(w, z, w_0, z_0). \tag{3.18}$$

Here $\chi(w, z, w_0, z_0)$ is a certain function on the set $\{w \in \gamma, z \in E, w_0 \in \gamma, z_0 \in \gamma'\}$. Take the integral

$$I_2(w_0, z_0) = -\frac{1}{4\pi^2} \sum_{j=1}^{\nu} \int_{\sigma'_j} \frac{\Psi_j}{w - w_0} dw \wedge dz.$$

The difference $I_1 - I_2$ is holomorphic in the bicylindrical domain $\gamma \times \gamma'$ by identity (3.18). Therefore it suffices to investigate the integral I_2 instead of the integral I_1. Putting $w = w_0$ in relation (3.12), we find that

$$Z_j(w_0, z) - Z_j(w_0, z_0) = (z - z_0) Q(w_0, z, w_0, z_0).$$

Hence it follows from relation (3.17) that

$$I_2(w_0, z_0) = -\frac{1}{4\pi^2}\sum_{j=1}^{\nu}\int_{\sigma'_j}\frac{\varphi(w_0, z)}{(w-w_0)(z-z_0)}\,dw\wedge dz. \tag{3.19}$$

We calculate the integral (3.19) by successive integration with respect to w (i.e., to u) and to z. Let l be the diameter of the disk γ lying in the u-axis and let u' be an arbitrary point of l. Let $\Gamma_{u'}$ and $\Gamma^j_{u'}$ be projections on the z-plane of the sections of σ and σ_j by the plane $w = u'$. Evidently $|Z_j(u', z)| = 1$ for $z \in \Gamma^j_{u'}$.

TWO POSSIBLE CASES: 1) The last relation is realized identically (due to the substitution $w = u'$); in this case $\Gamma^j_{u'}$ consists of a finite set of points (since then $\partial Z_j/\partial z$ vanishes at all points of $\Gamma^j_{u'}$); 2) the relation $|Z_j(u', z)| = 1$ is an equation and determines a finite set of analytic curves in the plane (and possibly, in addition, some isolated points). In the calculation of integral (3.19) only the second case can have a value. The intersection $\Sigma_j \cap \Sigma_k$ is not more than one-dimensional. Therefore to the set $\{|Z_j(u', z)| = 1,\ |Z_k(u', z)| = 1\}$ on the z-plane there belong only a finite set of points (with the possible exception of isolated values u' on l). We assume that the point u' that we have chosen is not exceptional (if exceptional points exist, they should be removed by being cut out from l; the integral (3.19) should then be regarded as improper); then the set $\Gamma^j_{u'} \cap \Gamma^k_{u'}$ does not contain any arc; $\Gamma_{u'}$ turns out to be the sum of $\Gamma^j_{u'}$.

Consider a small arc s lying on one of the curves $\Gamma^j_{u'}$; say, on $\Gamma^1_{u'}$, for definiteness. We assume that $\partial Z_1(u', z)/\partial z$ does not vanish on s (for this purpose we may, perhaps, need to avoid the case when certain points of $\Gamma^1_{u'}$ belong to s). We have $|Z_1(u', z)| < 1$ on one side of s and $|Z_j(u', s)| \leq 1$. The equality can hold only on a finite set of points of the curve $\Gamma^1_{u'}$. We shall avoid the case when these belong to s; therefore we exclude the possibility of such an equality on s.

The space arc $\{w = u',\ z \in s\}$ corresponding to the arc s will then be situated on a piece of an analytic hypersurface belonging to the boundary of some connected component of the set $\{|Z_j(w, z) < 1,\ (w, z) \in D_3,\ j = 1, \cdots, \nu\}$ and forming part of Δ. Such arcs together form $\Gamma^1_{u'}$; thus $\Gamma_{u'}$ consists of a series of closed curves (in addition, there may still belong to $\Gamma_{u'}$ some set of isolated points; they have no significance for the calculation of integral (3.19)). Let $(\Gamma_{u'})$ be the part of the z-plane bounded by the curve $\Gamma_{u'}$.

We take the bicylinder $\gamma \times \gamma'$ sufficiently small to ensure that γ' belongs to $\Gamma_{u'}$ and that the function $\phi(w_0, z)$ is holomorphic on the set $\{z \in \Gamma_{u'}, w_0 \in \gamma, z_0 \in \Gamma_{u'}\}$.

Let (w_0, z_0) be a point of $\gamma \times \gamma'$ not belonging to S, and let w be a point of l not belonging to the set of exceptional points. Then in view of Cauchy's integral formula

$$\frac{1}{2\pi i} \int\limits_{\Gamma_{u'}} \frac{\varphi(w_0, z)}{(w - w_0)(z - z_0)} dz = \frac{\varphi(w_0, z_0)}{w - w_0}. \tag{3.20}$$

Hence it further follows that

$$I_2(w_0, z_0) = \frac{1}{2\pi i} \int\limits_{l} \frac{\varphi(w_0, z_0)}{w - w_0} dw. \tag{3.21}$$

This integral is taken along the line l in the direction of increasing u. In order to make equality (3.21) obvious, it suffices to replace the double integral (3.19) by an iterated integral and to substitute for the inner integral its value from (3.20). We now see that integral (3.19), and therefore integral (3.14), show the required discontinuity on the passage through S. Our lemma is proved.

Now we shall aim to construct holomorphic functions in Δ_1 and Δ_2 which differ from each other at the points of S by a preassigned function $\phi(w, z)$. For this purpose it is sufficient to make a slight change in the integral (3.14). Consider the domain of integration S. It is situated on the pieces of the real analytic manifold $\Sigma_j = \{v = 0, |Z_j(w, z)| = 1, |Z_k(w, z)| \leq 1, j \neq k\}$. They are limits from the outside of open sets consisting of domains of holomorphy.[1] Let $V_j \supseteq \sigma_j$ be one of the open sets entering into this sequence. We are interested in finding the function $\Phi_j(w, z, w_0, z_0)$ meromorphic for $(w, z) \in V_j$ and $(w_0, z_0) \in D_1$, which has the same poles as the function $\psi_j(w, z, w_0, z_0)$ from Lemma 3 for $(w_0, z_0) \in D_3$ and is holomorphic for $(w_0, z_0) \in D_1 \setminus D_3$. The open set $\{(w, z) \in V_j, (w_0, z_0) \in D_1\}$ consists of domains of holomorphy; therefore it is possible to solve the problem just formulated (this is Cousin's first problem).

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1) Such a sequence can be constructed, for example, by means of the open sets $\{|v_n| < \epsilon_n, 1 - \epsilon_n < |Z_j(w, z)| < 1 + \epsilon_n, |Z_k(w, z)| < 1 + \epsilon_n, k \neq j\}$ (where $\epsilon_n \to 0$). Then it is easy to establish on the basis of the criterion constantly used that these open sets consist of domains of holomorphy (see, for example, the proof that the open set Δ_3 consists of domains of holomorphy).

The function $\Phi_j - \psi_j$ will be holomorphic on the open set $\{(w, z) \in V_j, (w_0, z_0) \in D_3\}$. The open set D_3 consists of domains of holomorphy. Hence *for every* $\epsilon > 0$ *and any open set* $V'_j \times D'_3$ *included together with its boundary in the interior of the set* $V_j \times D_3$ *one can find a function* $F_j(w, z, w_0, z_0)$, *holomorphic in* $V_j \times D_1$, *such that on the open set* $V'_j \times D'_3$

$$|(\Phi_j - \psi_j) - F_j| < \epsilon. \tag{3.22}$$

The set $V'_j \times D'_3$ *is taken here in such a way that* $\overline{\{\sigma_j, S\}} \subset V'_j \times D'_3$.

PROOF. We construct open sets V''_j and D''_3 in such a way that $V'_j \Subset V''_j \Subset V_j$ and $D'_3 \Subset D''_3 \Subset D_3 \subset D_1$. Since V_j and D_1 consist of domains of holomorphy, one can find for any points $M \in \partial V''_j$ and $N \in \partial D''_3$ neighborhoods γ_M and δ_N and functions $f_1(w, z)$ (holomorphic in V_j) and $f_2(w_0, z_0)$ (holomorphic in D_1) such that $\sup |f_1(V'_j)| < 1 - \epsilon$, $\sup |f_2(D'_3)| < 1 - \epsilon$, while $|f_1(w, z)| > 1$ at the points of γ_M and $|f_2(w_0, z_0)| > 1$ at the points of δ_N.

Let points $M_1, \cdots, M_\mu$ and $N_1, \cdots, N_\lambda$ be so chosen that their neighborhoods $\gamma_1, \cdots, \gamma_\mu$ and $\delta_1, \cdots, \delta_\lambda$ cover the boundaries $\partial V''_j$ and $\partial D''_3$, while functions $S_k(w, z)$, $k = 1, \cdots, \mu$ and $T_l(w_0, z_0)$, $l = 1, \cdots, \lambda$ play for them the roles of the functions $f_1(w, z)$ and $f_2(w_0, z_0)$.

We consider the open sets $\{|S_k(w, z)| < 1,\ k = 1, \cdots, \mu;\ (w, z) \in V_j\}$ and $\{|T_l(w_0, z_0)| < 1,\ l = 1, \cdots, \lambda;\ (w_0, z_0) \in D_1\}$. In view of what has been said we can single out parts of their connected components (we call them the parts E_j and $\mathfrak{E}$, respectively) such that we have

$$V'_j \Subset E_j \subset V_j; \quad D'_3 \Subset \mathfrak{E} \Subset D_3 \subset D_1,$$

$$V'_j \times D'_3 \Subset E_j \times \mathfrak{E} \subset V_j \times D_3 \subset V_j \times D_1.$$

The function $\Phi_j - \psi_j$ is holomorphic on the open set $V_j \times D_3$ and, accordingly, also on the open set $E_j \times \mathfrak{E}$. Therefore by Lemma 9 in §2, Chapter I it may be uniformly approximated on the set $V'_j \times D'_3$ by polynomials in w, z, w_0, z_0, S_k, T_l ($k = 1, \cdots, \mu$; $l = 1, \cdots, \lambda$), i.e., by functions holomorphic in $V_j \times D_1$. Our assertion is proved.

Thus on the open set $V_j \times D_3$ we have defined the functions $A_j = \Phi_j - \psi_j - F_j$ ($j = 1, \cdots, \nu$) satisfying the condition (3.22). Reasoning similarly for the open

set D_2 we define analogous functions B_j $(j = 1, \cdots, \nu)$ on the same open set $V_j \times D_3$. By means of these functions we construct the integrals

$$I_1(w_0, z_0) = -\frac{1}{4\pi^2} \sum_{j=1}^{\nu} \int_{\sigma_j} (\psi_j + A_j) \varphi^1(w, z)\, dw \wedge dz; \tag{3.23}$$

$$I_2(w_0, z_0) = -\frac{1}{4\pi^2} \sum_{j=1}^{\nu} \int_{\sigma_j} (\psi_j + B_j) \varphi^1(w, z)\, dw \wedge dz. \tag{3.24}$$

Here $\phi^1(w, z)$ is some function holomorphic in the neighborhood of S. Integrals (3.23) and (3.24) are to be taken over the outer side of σ. When the points $(w, z) \in \sigma_j$, $(w_0, z_0) \in D_3$, the function $A_j + \psi_j$ is equal to $\Phi_j - F_j$. The latter is defined on the whole set $V_j \times D_1$ and is holomorphic for $(w_0, z_0) \in \Delta_1$. Hence it follows that the function $I_1(w_0, z_0)$ is holomorphic for $(w_0, z_0) \in \Delta_1$. Analogously it may be established that the function $I_2(w_0, z_0)$ is holomorphic for $(w_0, z_0) \in \Delta_2$.

For points lying in the neighborhood of S, we have from relation (3.21)

$$I_1(w_0, z_0) - I_2(w_0, z_0)$$
$$= \varphi^1(w_0, z_0) - \frac{1}{4\pi^2} \sum_{j=1}^{\nu} \int_{\sigma_j} (A_j - B_j) \varphi^1(w, z)\, dw \wedge dz.$$

Now we can prove the

LEMMA 4. *Let the function $\phi(w, z)$ be holomorphic in $\widetilde{S}$ ($\widetilde{S}$ is the neighborhood of S). Then there exists a function $\phi^1(w, z)$ holomorphic in $\widetilde{S}$ such that*

1) *integral* (3.23) *defines the holomorphic function $I_1(w_0, z_0)$ in the domain $\Delta_1 + \widetilde{S}$, while integral* (3.24) *defines the holomorphic function $I_2(w_0, z_0)$ in the domain $\Delta_2 + \widetilde{S}$;*

2) *at the points of $\widetilde{S}$*

$$I_1(w_0, z_0) - I_2(w_0, z_0) = \varphi(w_0, z_0). \tag{3.25}$$

PROOF. We shall regard (3.25) as the equation for the function $\phi^1(w, z)$. We rewrite this equation in the following form:

$$\varphi^1(w_0, z_0) = \lambda \sum_{j=1}^{\nu} \int_{\sigma_j} K_j(w, z, w_0, z_0) \varphi^1(w, z)\, dw \wedge dz$$
$$+ \varphi(w_0, z_0), \tag{3.26}$$

where

$$K_j(w,\ z,\ w_0,\ z_0) = \frac{1}{4\pi^2}(A_j - B_j); \quad \lambda = 1.$$

We must find a solution of this integral equation that is holomorphic in $\widetilde{S}$. We shall apply the method of successive approximation. Set formally

$$\varphi^1(w_0,\ z_0) = \varphi_0^1(w_0,\ z_0) + \lambda\varphi_1^1(w_0,\ z_0) + \lambda^2\varphi_2^1(w_0,\ z_0) + \cdots. \tag{3.27}$$

Here λ is a complex parameter. Substituting the series (3.27) into equation (3.26), we arrive at the relations

$$\varphi_0^1(w_0,\ z_0) = \varphi(w_0,\ z_0),\quad \varphi_1^1 = K(\varphi_0^1),\quad \varphi_2^1 = K(\varphi_1^1),\ \ldots. \tag{3.28}$$

Here

$$K(\varphi_p^1) = \sum_{j=1}^{\nu} \int_{\sigma_j} K_j(w,\ z,\ w_0,\ z_0)\varphi_p^1(w,\ z)\,dw \wedge dz. \tag{3.29}$$

The functions $\phi_p^1(w_0,\ z_0)$ defined successively by the equalities (3.28) are holomorphic in $\widetilde{S}$ since the functions $\phi(w_0,\ z_0)$ and $K_j(w,\ z,\ w_0,\ z_0)$ are holomorphic there for $(w,\ z) \in \sigma_j$. Substitute the functions $\phi_p^1(w_0,\ z_0)$ thus obtained into the series (3.27). It remains only to establish the convergence of the series in the disk $|\lambda| < 1 + \epsilon'$ $(\epsilon' > 0)$. Let U be an open set taken in such a way that $D_3' \Supset U \Supset S$ and the function $\phi(w_0,\ z_0)$ is holomorphic on U; let $|\phi_p^1| \le M_p$ for $(w_0,\ z_0) \in U$ (here M_p is a certain positive number). Finally set $N = \sum_{j=1}^{\nu}\int_{\sigma_j} d\sigma$. Then in view of (3.29) it turns out that

$$|K(\varphi_p^1)| \leqslant \frac{\varepsilon}{2\pi^2} M_p N; \qquad p = 0,\ 1,\ 2,\ \ldots$$

Hence it follows that since $|A_j| < \epsilon$, $|B_j| < \epsilon$ for all j, we have, by (3.22)

$$M_p \leqslant \left(\frac{\varepsilon N}{2\pi^2}\right)^p M_0.$$

Evidently it is sufficient to take $\epsilon < N/2\pi^2(1 + \epsilon')$ in order to ensure the convergence of the series (3.27). Therefore our lemma is proved.

4. **Completion of the proof of Oka's principle.** Now we can pass to the proof of the last auxiliary proposition.

LEMMA 5. *The domain Δ is a domain of holomorphy.*

PROOF. Consider the open set

$$\Delta_1^{(\epsilon)} = \Delta \cap \{v > -\epsilon\}.$$

Here ϵ is a small positive number. We shall show that $\Delta_1^{(\epsilon)}$ consists of a domain of holomorphy. Indeed, it lies in G_1 (as a result of the smallness of ϵ); its boundary points belong 1) to the boundary of G_1 or 2) to the hyperplane $L^{(\epsilon)} = \{v = -\epsilon\}$, or 3) to one of the hypersurfaces $|Z_j(w, z)| = 1$. Let M be a boundary point of $\Delta_1^{(\epsilon)}$ of the third type lying (for definiteness) on the hypersurface $|Z_1(w, z)| = 1$. We construct in G_1 a meromorphic function $\Phi(w, z)$, equivalent in G_3 (in the sense of subtraction) to the function $[Z_1(w, z) - Z_1(M)]^{-1}$ and holomorphic on the part of the open set G_1 not entering into G_3. In view of the first assertion of Lemma 2, $|Z_j(w, z)| < 1$ in the neighborhoods of L_1 and L_2; there the function $\Phi(w, z)$ is holomorphic everywhere it is defined.[1] For the boundary points of $\Delta_1^{(\epsilon)}$ of the first and second types we can also construct functions that are holomorphic in $\Delta_1^{(\epsilon)}$ and have these points as singularities. This follows from the fact that G_1 consists of domains of holomorphy (see the proof of Lemma 2), while $L^{(\epsilon)}$ is a hyperplane. What we have said is sufficient to show on the basis of the usual criterion that the open set $\Delta_1^{(\epsilon)}$ consists of domains of holomorphy. Analogous conclusions may also be drawn for the open set

$$\Delta_2^{(\epsilon)} = \Delta \cap \{v < \epsilon\}.$$

We now turn to the discussion of the domain Δ. Assume that Cousin's first problem is set up in a certain way in Δ. Then this problem is solvable for the open sets $\Delta_1^{(\epsilon)}$ and $\Delta_2^{(\epsilon)}$, since these sets consist of domains of holomorphy. Let meromorphic functions $\Phi_1(w, z)$ and $\Phi_2(w, z)$ be such solutions on the sets $\Delta_1^{(\epsilon)}$ and $\Delta_2^{(\epsilon)}$, respectively. At the points of the set S these functions are equivalent to each other with respect to subtraction and their difference $\phi(w, z) = \Phi_1(w, z) - \Phi_2(w, z)$ will be holomorphic. For this function $\phi(w, z)$ we find on Δ_1 and Δ_2, respectively, the functions $I_1(w, z)$ and $I_2(w, z)$ by using integrals (3.23) and (3.24), such that $I_1(w, z) - I_2(w, z) = \phi(w, z)$ for $(w, z) \in S$. Then the equalities

$$\Phi(w, z) = \begin{cases} \Phi_1(w, z) - I_1(w, z) & \text{for} \quad (w, z) \in \Delta_1, \\ \Phi_2(w, z) - I_2(w, z) & \text{for} \quad (w, z) \in \Delta_2 \end{cases}$$

define the meromorphic function $\Phi(w, z)$ in the domain Δ. For values of the

1) All within the limits of G_1. As mentioned above, we have changed the notations: we shall speak of the hyperplanes $v = a_1$, $v = a_2$ instead of the hyperplanes $v = b_1$, $v = b_2$ and of L_1 and L_2 instead of L_1' and L_2'; therefore the open sets G_1 and G_2 now extend to L_1 and L_2 (not to L_1' and L_2').

function $\Phi(w, z)$ at the points of the set S one may use the limiting values of the functions $\Phi_1 - I_1$ and $\Phi_2 - I_2$ (which coincide with each other). This function is the solution of the general first problem of Cousin formulated in the domain Δ. The domain in which Cousin's first problem is always solvable is a domain of holomorphy in view of Theorem 7.2. The lemma is proved.

From the third assertion of Lemma 2 it follows that the domain D may be approximated from inside by the domains Δ. Hence, by Theorem 6.1, it follows that D is a domain of holomorphy. Theorem 14.1 is proved.

5. **Domains convex in the sense of H. Cartan.** The following condition for the convexity of a domain in the sense of H. Cartan (see §12.8, Chapter II, (I)) can serve as a link intermediate between the condition for the holomorphic convexity of a domain (which is necessary and sufficient that it should be a domain of holomorphy) and the condition for the analytic convexity of a domain in the sense of Hartogs.

A domain $D \subset C^2_{w,z}$ is said to be analytically convex in the sense of H. Cartan if every finite point $P \in \partial D$ can be enclosed by a hyperball V such that the open set $D \cap V$ will consist of domains of holomorphy.

THEOREM 14.2. *The domain $D \subset C^2_{w,z}$ convex in the sense of H. Cartan is a domain of holomorphy.*

PROOF. We shall carry out the proof for the case of a bounded domain D. We divide the plane C^1_w into equal rectangles ω with sides parallel to the coordinate axes, and the plane C^1_z into rectangles ω' (of the same type). Then the whole space $C^2_{w,z}$ is divided into bicylindrical domains $\omega \times \omega'$. The parts of the domain D contained in each of the domains $\omega \times \omega'$ consist of domains of holomorphy. This follows from the fact that 1) the domains $\omega \times \omega'$ are themselves domains of holomorphy (they may belong completely to a domain); 2) the open domains appearing as a result of the intersection of the domain D with each of the domains $\omega \times \omega'$ consist of domains of holomorphy. The latter result in turn is clear from the fact that 1) the domain D is convex in the sense of H. Cartan, 2) the intersection of domains of holomorphy consist of domains of holomorphy.

The proof is completed by the application of Theorem 14.1.

Now we can prove the

THEOREM 14.3 (Oka). *The domain $D \subset C^2_{w,z}$ analytically convex in the sense of Hartogs is a domain of holomorphy.*

We shall carry out the proof of this theorem for the case of a bounded domain. We preface it by the deduction of a series of lemmas.

LEMMA 6. *If a function* $\phi(w, z)$ *belongs to the class* $\mathcal{C}^2$ *and is plurisubharmonic in a domain* $D \subset C^2_{w,z}$, *and if* $H(\phi) > 0$, *then there passes through each point* $P \in D$ *an analytic surface on which* $\phi(w, z) > \phi(P)$ *within the limits of a certain deleted neighborhood of the point* P.

PROOF. First assume that $(\operatorname{grad}\phi)_P \neq 0$. Then, as was established at the beginning of §13.5, $[L(\phi)] > 0$ under the conditions of the lemma. Here $L(\phi)$ is Levi's determinant. Therefore, by Levi's Theorem 12.7, (I), in the case at hand there indeed passes through the point P an analytic surface $f(w, z) = 0$ (here as the function $f(w, z)$ one can always take a polynomial of second degree) such that on this surface $\phi(w, z) > \phi(P)$ in some deleted neighborhood of the point P.

We further investigate the case when $(\operatorname{grad}\phi)_P = 0$. For simplicity we place the origin of coordinates at the point P and assume that $\phi(P) = 0$. Next we take the analytic plane $w = \alpha t$, $z = \beta t$ (α, β are complex parameters) and consider the function $\phi(w, z) = \Phi(t)$ on that plane. Since the function ϕ has continuous partial derivatives up to the second order inclusive, we shall have in the neighborhood of the point P

$$\Phi = (\mu t^2 + 2\nu t\bar{t} + \overline{\mu t}^2) + (\epsilon t^2 + 2\delta t\bar{t} + \overline{\epsilon t}^2).$$

Here μ is a complex quantity and ν a real quantity; these two depend on α, β; ϵ, δ and in addition, depend on t and tend to zero together with it. We shall show that in our case α and β may be so chosen that $\Phi > 0$ for all $|t|$ not exceeding a certain quantity. To this end it suffices to establish that $\nu > 0$ and $\nu > |\mu|$. We have

$$\nu = \phi''_{w\bar{w}}\alpha\bar{\alpha} + \phi''_{w\bar{z}}\alpha\bar{\beta} + \phi''_{\bar{w}z}\bar{\alpha}\beta + \phi''_{z\bar{z}}\beta\bar{\beta},$$

$$\mu = \phi''_{w^2}\alpha^2 + 2\phi''_{wz}\alpha\beta + \phi''_{z^2}\beta^2.$$

Here $\nu = [H(\phi)]_P$ is always larger than zero. It remains only to choose α and β so that $\mu = 0$; this is always possible. Thus Lemma 6 is proved.

LEMMA 7. *If a function* $\phi(w, z)$ *belongs to the class* $\mathcal{C}^2$ *and is plurisubharmonic in a domain* $D \subset C^2_{w,z}$, *and if* $H(\phi) > 0$ *and if* S *is a hyperball with its center at a point* $P \in D$ *and* $S \subset D$, *then the open set* $\Sigma = \{\phi(w, z) < \phi(P)\} \cap S$ *consists of domains of holomorphy.*

PROOF. Let the domain Δ be one of the connected components of the set Σ. We consider the holomorphy hull of Δ, the domain $H(\Delta)$. Assume that $H(\Delta) \neq \Delta$. Because of the method of defining the domain Δ, the domain $H(\Delta)$ contains at least one point at which the function ϕ takes on the value $\phi(P)$. Let α be the upper bound of the values assumed by the function ϕ in the domain $H(\Delta)$. Then $\phi(P) < \alpha$ in view of Theorem 13.2 (4).

Let Q be a boundary point of the domain $H(\Delta)$ and let $\phi(Q) = \alpha$. Then there exists an analytic surface $f(w, z) = 0$, passing through Q and remaining entirely within the limits of the neighborhood of the point Q, for which $\phi(w, z) > \phi(Q) = \alpha$, i.e., outside the domain $H(\Delta)$.

We now form a hyperball δ with its center at the point Q. We take the radius of this hyperball so small that 1) the function $f(w, z)$ will be holomorphic in the closed hyperball $\bar{\delta}$; 2) the intersection of the hypersphere $\partial\delta$ with the surface $f(w, z) = 0$ will be situated outside Δ. Evidently such a hyperball δ necessarily exists.

Next we take a small number $\epsilon > 0$ such that the intersection of the hypersurface $|f| = \epsilon$ with the hypersphere $\partial\delta$ will also be situated outside the domain $H(\Delta)$.

Consider the set E of the points of the domain $H(\Delta)$ either situated outside the closed hyperball $\bar{\delta}$, or of those points for which $|f| > \epsilon$. Then E is an open set containing the domain Δ. Each connected component of the set E is *convex in the sense of H. Cartan* and therefore is a domain of holomorphy. This contradicts the method of defining the domain $H(\Delta)$. Therefore $H(\Delta) = \Delta$; our lemma is proved.

COROLLARY 1. *The domain bounded by the hypersurface* $\phi(w, z) = 0$, *where the function* $\phi(w, z)$ *belongs to the class* $\mathcal{C}^2$ *and satisfies the condition* $H(\phi)>0$ (it should possess all of these properties only in some neighborhood of the boundary of the domain; here it is assumed that the values $\phi < 0$ correspond to the inside of the domain), *is convex in the sense of H. Cartan.*

Hence by using Theorem 13.2 we obtain the

COROLLARY 2. *The bounded domain with boundary* $\phi(w, z) = 0$, *where the function* $\phi(w, z)$ *belongs to the class* $\mathcal{C}^2$ *and satisfies the condition* $H(\phi) > 0$ (it should have these properties only in some neighborhood of the boundary; here it is assumed that the values $\phi < 0$ correspond to the inside of the domain), *is a domain of holomorphy.*

COMPLETION OF THE PROOF OF THEOREM 14.3. The domain D analytically convex in the sense of Hartogs may be approximated by the domains $\{-\ln d_D(w, z) < \alpha\}$ for $\alpha \longrightarrow \infty$ (see the corollary of Theorem 13.4); here $\{-\ln d_D(w, z)\}$ is a continuous plurisubharmonic function in the domain D (see Theorem 13.4 and its complement). In view of Theorem 13.9 this function in turn may be approximated by a plurisubharmonic function $\psi(w, z)$ having the properties used in the last lemma. We take three numbers $\alpha < \beta < \gamma$ such that the set of those points of D for which $-\ln d_D(w, z) < \alpha$ turns out to be nonvoid. We select a domain D_0 in such a way that $\{-\ln d_D(w, z) < \alpha\} \subset D_0 \Subset D$. Assume that $|\psi(w, z) - [-\ln d_D(w, z)]| < \epsilon$ for $(w, z) \in D_0$. Here ϵ is to be taken so small that

$$\{-\ln d_D(w, z) < \alpha\} \subset \{\psi(w, z) < \beta\} \subset \{-\ln d_D(w, z) < \gamma\}.$$

By Corollary 2 of Lemma 7 the open set $\{\psi(w, z) < \beta\}$ consists of domains of holomorphy. When $\alpha \longrightarrow \infty$, the open set $\{-\ln d_D(w, z) < \alpha\}$, that is, the open set $\{\psi(w, z) < \beta\}$ also tends to the domain D. Thus D turns out to be a limit of a sequence of domains of holomorphy. Hence (since D is a bounded domain) by the Behnke-Stein Theorem 6.3 we can conclude that D is a domain of holomorphy.

REMARK 1. The Theorem 14.3 just proved is valid not only for a bounded domain, but also for any finite domain. Its proof (in this case) is based on Behnke-Stein's generalized theorem which also covers the case of an arbitrary finite domain.

REMARK 2. From Theorem 14.3 and the results of §13.5 it follows that:

The domain bounded by the hypersurface $\phi(w, z) = 0$, where the function $\phi(w, z)$ belongs to the class $\mathcal{C}^2$ (the values $\phi < 0$ correspond to the inside of the domain) and satisfies the condition $L(\phi) \geq 0$, is a domain of holomorphy.

As we have already remarked, together with Theorem 14.3 we also have the

THEOREM 14.4 (Bremermann, Norguet, Oka). *The domain $D \subset C^n$ analytically convex in the sense of Hartogs is a domain of holomorphy.*

§15. THE ŠILOV AND BERGMAN BOUNDARIES OF DOMAINS OF HOLOMORPHY

1. The boundary of an open set in the sense of G. E. Šilov.

DEFINITION. Let $D \subset C_z^n$ be an open set, and let $\mathfrak{S} = \{\phi(z)\}$ be a collection of real or complex functions upper semicontinuous on the closed set $\overline{D}$. A closed set $S_{\mathfrak{S}}(D) \subset \overline{D}$ is called *the Šilov boundary of the domain D relative to the collection* $\mathfrak{S}$, if

1r) for any function $\phi \in \mathfrak{S}$

$$\sup_{z \in \bar{D}} \varphi(z) = \sup_{z \in S_{\mathfrak{S}}(D)} \varphi(z)$$

(when the collection $\mathfrak{S}$ consists of real functions):

1c) for any function $\phi \in \mathfrak{S}$

$$\sup_{z \in \bar{D}} |\varphi(z)| = \sup_{z \in S_{\mathfrak{S}}(D)} |\varphi(z)|$$

(when the collection $\mathfrak{S}$ consists of complex functions);

2) every other closed subset $S_1 \subset \bar{D}$ having the above-stated property contains the set $S_{\mathfrak{S}}(D)$.

This definition is extended without any change to the open subset of an arbitrary complex space.

In connection with this definition we recall that a complex function $\phi(z)$ is said to be upper semicontinuous if $|\phi(z)|$ is upper semicontinuous.

If $\mathfrak{S}$ is the ring of all functions holomorphic on the set D and continuous on the closed set $\bar{D}$, then in view of the maximum principle $S_{\mathfrak{S}}(D) \subset \partial D$. For conciseness this set $S_{\mathfrak{S}}(D)$ will simply be called the *Šilov boundary of the set* D and will be denoted by the symbol $S(D)$.

If in the definition of the Šilov boundary the collection of all functions holomorphic on the closed set $\bar{D}$ is taken as $\mathfrak{S}$, one obtains *the boundary of the set* D *in the sense of Bergman* [4] denoted by the symbol $B(D)$; if the collection of polynomials is taken as $\mathfrak{S}$, one obtains the *polynomial boundary* of the set D denoted by the symbol $P(D)$. Evidently $P(D) \subset B(D) \subset S(D) \subset \partial D$.

These boundaries, generally speaking, are distinct. Later (see Example 1 of Theorem 16.4 in §16.3) we shall present an example, constructed by L. A. Aĭzenberg, of a domain D for which $B(D) \neq S(D)$.

The following concept will be useful in the sequel:

The collection $\mathfrak{S}$ of real upper semicontinuous functions on the compactum $\bar{D} \subset C_z^n$ is said to be *proper* on the set D if $\phi(z) \in \mathfrak{S}$ implies that all the functions of the form $\phi(z) + \ln |z_i - c| \in \mathfrak{S}$. Here $i = 1, \cdots, n$, and c is an arbitrary complex number.

A closed set $\Gamma \subset \bar{D}$ is said to be *determining* for the collection $\mathfrak{S}$ if each function $\phi(z) \in \mathfrak{S}$ takes on in that set the maximum value for the whole set D.

Thus $S_{\mathfrak{S}}(D)$ is the minimum determining set for the collection $\mathfrak{S}$.

THEOREM 15.1. *The proper collection* $\mathfrak{S}$ *of the functions defined on a compactum* $\bar{D} \subset C_z^n$ *always has the uniquely defined boundary* $S_{\mathfrak{S}}(D)$.[1)]

PROOF. I. The set $\bar{D}$ itself is determining for the collection $\mathfrak{S}$ since all the functions $\phi(z) \in \mathfrak{S}$, in view of the upper semicontinuity, attain their maximum value on this set $\bar{D}$. If the set $\Gamma_1 = \bar{D}$ is not the minimum determining set, then there exists a determining set $\Gamma_2 \subset \Gamma_1$; if the set Γ_2 is not minimal, then there exists a determining set $\Gamma_3 \subset \Gamma_2$ and so forth. Thus there is defined the collection of sets $\{\Gamma_i,\ i \in I\}$, where I is a certain ordered set of indices and $\Gamma_{i_1} \supset \Gamma_{i_2}$ for $i_2 \succ i_1$. In view of the compactness of the set $\bar{D}$ the sets Γ_i have a nonempty intersection, which turns out to be the minimum determining set.

II. Let us assume * that there exist two distinct Šilov boundaries $S^1_{\mathfrak{S}}(D)$ and $S^2_{\mathfrak{S}}(D)$. Then there exist

1) a point z^1 with

$$z^1 = (z_1^1, \cdots, z_n^1) \in S^1_{\mathfrak{S}}(D) \setminus S^2_{\mathfrak{S}}(D)$$

and a number $\epsilon > 0$ such that $U(z^1, \epsilon) \cap S^2_{\mathfrak{S}}(D) = \emptyset$, where $U(z^1, \epsilon) = U$ is the polycylinder of radius ϵ with center at the point z^1; and

2) a function $\phi(z) \in \mathfrak{S}$ attaining its maximum m on $S^1_{\mathfrak{S}}(D)$ within the limits of the intersection $U \cap S^1_{\mathfrak{S}}(D)$ and remaining smaller than m on $S^1_{\mathfrak{S}}(D) \setminus D$, since $S^1_{\mathfrak{S}}(D)$ is the minimum determining set.

If we set $m^* = \sup \phi(z)$ for $z \in S^1_{\mathfrak{S}}(D) \setminus U$, it follows from the above that $m^* < m$.

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1) In the proof of Theorem 15.1 one uses the idea of the proof of the related theorem of G. E. Šilov. See I. M. Gel'fand, D. A. Raĭkov and G. E. Šilov, *Commutative normed rings*, Fizmatgiz, Moscow, 1960, pp. 76–77 (Russian).

Theorems 15.1, 15.2 and their corollaries are due to L. A. Aĭzenberg. For the proof of existence and uniqueness of the Šilov boundary in the case of the multiplicative semigroup $\mathfrak{S}$, see R. Arens and I. M. Singer, *Function values as boundary integrals*, Proc. Amer. Math. Soc. 5 (1954), 735–745.

*Note by the editor of the translation. This proof of the uniqueness of the boundary is taken from a letter received from the author, to whom the Polish mathematician J. Siciak had pointed out a flaw in the proof given in the Russian text. Both the original proof and the correction are due to L. A. Aĭzenberg.

From our assumption also follows the existence of a point $z^2 = (z_1^2, \cdots, z_n^2) \in S_{\mathfrak{S}}^2(D) \setminus S_{\mathfrak{S}}^1(D)$ such that $\phi(z^2) = m$. Since $z^2 \notin U$, it follows that for some index i, with $1 \leq i \leq n$, we have $|z_i^2 - z_i^1| > \epsilon$. Let us set $R = \sup |z_i - z_i^2|$ for $z \in S_{\mathfrak{S}}^1(D)$. Finally, let us choose a number $\lambda > 1$ such that

$$\frac{m^* + \ln (R + \lambda |z_i^2 - z_i^1|)}{m + \ln (\lambda |z_i^2 - z_i^1|)} < 1 \tag{3.30}$$

and consider the function $\psi(z) = \phi(z) + \ln |z_i - z_i^2 + \lambda(z_i^2 - z_i^1)| \in \mathfrak{S}$. It is obvious that $\psi(z^2) = m + \ln (\lambda |z_i^2 - z_i^1|)$. For $z \in S_{\mathfrak{S}}^1(D) \setminus U$ it follows from (3.30) that

$$\psi(z) \leq m^* + \ln |z_i - z_i^2 + \lambda(z_i^2 - z_i^1)|$$
$$\leq m^* + \ln (R + \lambda |z_i^2 - z_i^1|) < m + \ln (\lambda |z_i^2 - z_i^1|).$$

For $z \in S_{\mathfrak{S}}^1(D) \cap U$ we have

$$\psi(z) \leq m + \ln |z_i - z_i^2 + \lambda(z_i^2 - z_i^1)|$$
$$= m + \ln |z_i - z_i^1 + (1 - \lambda)(z_i^1 - z_i^2)| < m + \ln (\lambda |z_i^2 - z_i^1|).$$

Consequently, for all $z \in S_{\mathfrak{S}}^1(D)$ it follows that $\psi(z) < \psi(z^2)$, which is impossible, since $S_{\mathfrak{S}}^1(D)$ is a Šilov boundary, and thus our assumption must be rejected.

COROLLARY 1. *Let D and $D^* \subset C^n$ be open sets, let the compactum $\bar{D} \subset D^*$ and let $\mathfrak{S}$ be the collection of plurisubharmonic (convex) functions on the set D^*. Then there exists a uniquely defined Šilov boundary $S_{\mathfrak{S}}(D)$ of the set D relative to the collection $\mathfrak{S}$.*

COROLLARY 2. *Let $D \subset C^n$ be a bounded open set, and let $\mathfrak{S}$ be the collection of functions plurisubharmonic (convex) on the open set D and upper semicontinuous on $\bar{D}$. Then there exists a uniquely defined Šilov boundary $S_{\mathfrak{S}}(D)$ of the set D relative to the collection $\mathfrak{S}$.*

THEOREM 15.2. *Let $D \subset C^n$ be a bounded open set, and let $\mathfrak{S}$ be a collection of complex functions (not reducing to a single function identically equal to zero) upper semicontinuous on the closure $\bar{D}$. If $\phi(z) \in \mathfrak{S}$ implies that $\phi(z)(z_i - c) \in \mathfrak{S}$, where c is an arbitrary complex number, then there exists a uniquely defined Šilov boundary $S_{\mathfrak{S}}(D)$ of the set D relative to the collection $\mathfrak{S}$.*

The proof is reduced to verifying that the collection $\{\ln |\phi(z)|\}$, where $\phi(z) \in \mathfrak{S}$, is proper on the compactum $\bar{D}$ and then applying Theorem 15.1.

COROLLARY. *Let $D \subset C^n$ be a bounded open set. Then there exist the uniquely defined:* 1) *Šilov boundary $S(D)$,* 2) *Bergman boundary $B(D)$,* 3) *polynomial*

boundary $P(D)$ *of that set.*

To obtain the first conclusion one needs to take as the collection $\mathfrak{S}$ in Theorem 15.2 the set of all functions holomorphic on the set D and continuous on its closure. Notice that this collection becomes a normed ring (in other words, a Banach algebra) if one adopts as the norm of the element $\phi(z) \in \mathfrak{S}$ the quantity $\max_{z \in \bar{D}} |\phi(z)|$. The boundary $S(D)$ represents the ring boundary of the space of the maximal ideals of this normed ring. The concept of a ring boundary was introduced by G. E. Šilov. He also proved its existence and uniqueness.[1)]

To obtain the second conclusion in the corollary one needs to take as the collection $\mathfrak{S}$ in Theorem 15.2 the set of all functions holomorphic on the closed set $\bar{D}$, and for the third conclusion, the set of all polynomials.

2. **The Šilov and Bergman boundaries for some classes of domains of holomorphy.**

THEOREM 15.3 (Bremermann [5]).[2)] *If a domain* $D \subset C^n$ *is bounded and strictly analytically convex in the sense of Hartogs, then*

$$B(D) = \partial D.$$

PROOF. By definition the domain D may be given by the condition

$$D = \{\phi(z) < 0\}.$$

Here $\phi(z)$ is a plurisubharmonic function in some neighborhood of the domain D and $H(\phi) > 0$ for $z \in \bar{D}$ and $\sum_{k=1}^{n} |\alpha_k|^2 > 0$ (see the definition of a domain strictly analytically convex in the sense of Hartogs in §13.6 of the present chapter). We shall restrict ourselves to considering the case when the function $\phi \in \mathcal{C}^2$. Let

$$\varphi^*(z) = \varphi(z) - \varepsilon \sum_{k=1}^{n} |z_k - z_k^0|^2,$$

where $\epsilon > 0$ is a certain number, and the point $z^0(z_1^0, \cdots, z_n^0) \in \partial D$. It is easy to see that

$$H\left(\varepsilon \sum_{k=1}^{n} |z_k - z_k^0|^2\right) = \varepsilon \sum_{k=1}^{n} |\alpha_k|^2 > 0.$$

Since $H(\phi) > 0$ for $z \in \bar{D}$ and $\sum_{k=1}^{n} |\alpha_k|^2 > 0$, and the function $\phi(z) \in \mathcal{C}^2$, there exists a number $\delta > 0$ such that

1) See the book referred to in the preceding footnote, pp. 75–81.

2) In the paper of Bremermann [5] it is proved that $S(D) = \partial D$ (see the subsequent corollary of Theorem 15.3). L. A. Aĭzenberg pointed out that there really follows from Bremermann's argument the stronger result: $B(D) = \partial D$. Similar strengthenings of the original formulation of Bremermann's Theorems 15.4, 15.5, 15.7 are also due to L. A. Aĭzenberg.

$$H(\varphi) > \delta \sum_{k=1}^{n} |\alpha_k|^2 \quad \text{for} \quad z \in \bar{D}.$$

We take the number $\epsilon < \delta$. Then $\phi^*(z^0) = 0$ and

$$H(\varphi^*) > 0 \quad \text{for} \quad z \in \bar{D}, \quad \sum_{k=1}^{n} |\alpha_k|^2 > 0;$$
$$\varphi^*(z) < 0 \quad \text{for} \quad z \in \bar{D} \setminus \{z_0\}.$$

We choose a number $c > 0$ such that

$$\psi(z) = c\varphi^*(z) < -2\varepsilon \quad \text{for} \quad z \in \bar{D} \setminus U_{z_0};$$

U_{z_0} is a neighborhood of the point z_0 chosen arbitrarily in advance.

Using Theorem 13.10 we can find among the functions $f_1(z), \cdots, f_k(z)$ appearing in this theorem a function $f_{j_0}(z)$, holomorphic in the closed domain $\bar{D}$, such that

$$\psi(z_0) - \varepsilon \leqslant c_{j_0} \ln |f_{j_0}(z_0)| \leqslant \psi(z_0),$$
$$c_{j_0} \ln |f_{j_0}(z)| \leqslant \psi(z) \quad \text{for} \quad z \in \bar{D}$$

(where c_{j_0} is a certain rational number) or

$$-\varepsilon \leqslant c_{j_0} \ln |f_{j_0}(z_0)| \leqslant 0,$$
$$c_{j_0} \ln |f_{j_0}(z)| < -2\varepsilon \quad \text{for} \quad z \in D \setminus U_{z_0}.$$

Thus it has been established that the function $c_{j_0} \ln |f_{j_0}(z)|$ attains its largest value in the closed domain $\bar{D}$ on the set $\partial D \cap U_{z_0}$. Consequently this set contains the boundary points of the domain D in the sense of Bergman. Since the neighborhood U_{z_0} may be chosen as small as desired, while the Bergman boundary $B(D)$ is a closed set, it follows that $z_0 \in B(D)$. Therefore, since z_0 is an arbitrary point of the boundary ∂D, our theorem is proved.

Since $B(D) \subset S(D)$, there follows from the theorem just proved the

COROLLARY. *If a domain $D \subset C^n$ is bounded and strictly analytically convex in the sense of Hartogs, then $S(D) = \partial D$.*

THEOREM 15.4. [1] *Let a Weil analytic polyhedron $\Delta = \{|Z_j(z)| < 1$, $j = 1, \cdots, N;\ z \in D\} \Subset D$ be defined in terms of the functions $Z_j(z)$, $j = 1, \cdots, N$,*

1) Theorem 15.4 is due to Bremermann [4]; the connection between the various kinds of boundaries of an analytic polyhedron was investigated in more detail in the article of Hoffman [1].

holomorphic in a domain $D \subset C^n$ *and having the property there that any set*

$$\Sigma(\theta_{j_1}, \ldots, \theta_{j_n}) = \left\{ Z_{j_1}(z) = e^{i\theta_{j_1}}, \ldots, Z_{j_n}(z) = e^{i\theta_{j_n}}, z \in D \right\},$$

(*where* $\theta_{j_1}, \cdots, \theta_{j_n}$ *are arbitrary constants,* $1 \le j_1 \le \cdots \le j_n < N$) *consists of a discrete set of points. Then*

$$S(\Delta) = B(\Delta) = S^*(\Delta),$$

where $S^*(\Delta) = \{|Z_j(z)| = 1,\ j = 1, \cdots, N;\ z \in D\}$ *is the skeleton of the polyhedron* Δ.

REMARK. The assumption on the set $\Sigma(\theta_{j_1}, \cdots, \theta_{j_n})$ will be realized if $\partial(Z_{j_1}, \cdots, Z_{j_n})/\partial(z_1, \cdots, z_n) \neq 0$ for $z \in D$.

The proof of Theorem 15.4 is based on the following lemma:

LEMMA 1. *Let* $\mathfrak{S}$ *be the collection of plurisubharmonic functions in some neighborhood* $\widetilde{\Delta}$ *of the polyhedron* Δ. *Then*

$$S_{\mathfrak{S}}(\Delta) \subset S^*(\Delta). \tag{3.31}$$

COROLLARY. *The inclusion relation* (3.31) *holds for the collection of moduli of functions holomorphic in the closed domain* Δ, *i.e.,*

$$B(\Delta) \subset S^*(\Delta). \tag{3.32}$$

PROOF OF LEMMA 1. As usual (see §22, Chapter IV, (I)), set

$$\sigma_j = \partial\Delta \cap \{|Z_j(z)| = 1\},$$

$$\sigma_{j_1 j_2} = \partial\Delta \cap \{|Z_{j_1}(z)| = 1, \quad |Z_{j_2}(z)| = 1\}$$

and so forth. By the last in this sequence we define the collection of surfaces $\sigma_{j_1 j_2 \cdots j_n}$. By our assumptions the function $Z_j(z) \neq$ const, and therefore the topological dimension of the surface $\{|Z_j(z)| = 1\} \cap D$ is equal to $2n - 1$. As usual, we assume that all the functions $Z_j(z)$ are essential for defining the polyhedron Δ. Therefore the surface σ_j has the same dimension. It is stratified into the analytic surfaces $\{Z_j(z) = e^{i\theta}\} \cap \partial\Delta$. By assumption the functions $Z_j(z)$ are holomorphic in the domain D, and accordingly the analytic surfaces defined by the equations $Z_j(z) = e^{i\theta}$ cannot have boundary points in the domain D. Hence it follows that in the case under consideration

$$\partial[\{Z_j(z)=e^{j\theta}\}\cap\partial\Delta]\subset\bigcup_{\rho\neq j}\sigma_{j\rho}.$$

In view of Theorem 13.3 the maximum principle for plurisubharmonic functions holds on the analytic surfaces. Therefore, if the function $\phi(z)$ is plurisubharmonic in the domain $\widetilde{\Delta}$ and $\phi(z)\leq M$ for $z\in\bigcup_{\rho\neq j}\sigma_{j_\rho}$, then $\phi(z)\leq M$ also for $z\in\sigma_j$. Here M is a certain real number. Then, applying this argument to the surfaces $\sigma_{j_1j_2}$, $\sigma_{j_1j_2j_3}$ and so forth, we ultimately arrive at the conclusion: if $\phi(z)\leq M$ for $z\in S^*(\Delta)$, then $\phi(z)\leq M$ for $z\in\partial\Delta$ and accordingly also for $z\in\overline{\Delta}$. With this our lemma is proved.

REMARK. For the application of Theorem 13.3 it is necessary that the corresponding surfaces should consist of ordinary points. We shall limit ourselves to this simplest case.

PROOF OF THEOREM 15.4. Let a point $z^0\in S^*$. Then, from the set of functions $Z_j(z)$, $j=1,\cdots,N$, one can choose $Z_{j_k}(z)$, $k=1,\cdots,n$, such that $Z_{j_k}(z^0)=e^{i\theta_{j_k}}$, $1\leq j_1<\cdots<j_n\leq N$. Set

$$g_k(z)=\frac{1}{2}\left(1+e^{-i\theta_{j_k}}Z_{j_k}(z)\right).$$

Then

$$g_k(z)=1\quad\text{for}\quad z\in\{Z_{j_k}(z)=e^{i\theta_{j_k}}\},$$
$$|g_k(z)|<1\quad\text{for}\quad z\in\overline{\Delta}\setminus\{Z_{j_k}(z)=e^{i\theta_{j_k}}\}.$$

Here $k=1,\cdots,n$. Next we form the function $\phi(z)=\Sigma_{k=1}^n\ln|g_k(z)|$. Evidently the function $\phi(z)$ is plurisubharmonic in the closed polyhedron $\overline{\Delta}$ and

$$\varphi(z)=0\quad\text{for}\quad z\in\Sigma_{z^0},$$
$$\varphi(z)<0\quad\text{for}\quad z\in\overline{\Delta}\setminus\Sigma_{z^0}.$$

Here $\Sigma_{z^0}=\{Z_{j_1}(z)=e^{i\theta_{j_1}},\cdots,Z_{j_n}(z)=e^{i\theta_{j_n}}\}$. In view of our assumption the intersection $\Sigma_{z^0}\cap S^*(\Delta)$ consists of a finite set of points $z^0,z^1,\cdots,z^m$.

Through each of the points z^ν, $\nu=1,\cdots,m$ we draw an analytic surface $\{l_\nu(z)=0\}$ of complex dimension $n=1$ which does not contain the point z^0. Here $l_\nu(z)=a_{\nu1}z_1+\cdots+a_{\nu n}z_n+a_{\nu0}$. Then the sum

$$\ln |l_1(z)| + \cdots + \ln |l_m(z)| \tag{3.33}$$

is equal to $-\infty$ at the points $z^1, \cdots, z^m$ and remains finite at the point z^0. We add to the quantity (3.33) a number c_0 such that the resulting expression will vanish at the point z^0. Finally we form the function

$$\varphi^*(z) = \sum_{k=1}^{n} c_k \ln |g_k(z)| + \sum_{p=1}^{m} \ln |l_p(z)| + c_0,$$

where c_k, $k = 1, \cdots, n$ are positive integers. These are to be chosen so large that the function $\phi^*(z)$ is plurisubharmonic in the closed polyhedron $\bar{\Delta}$ and

$$\begin{gathered} \phi^*(z^0) = 0, \\ \phi^*(z) < 0 \quad \text{for} \quad z \in \bar{\Delta} \backslash \{z^0\}. \end{gathered} \tag{3.34}$$

Since c_k, $k = 1, \cdots, n$ are positive integers, $e^{\phi^*(z)} = |h(z)|$, where

$$h(z) = e^{c_0} \prod_{k=1}^{n} [g_k(z)]^{c_k} \prod_{p=1}^{m} l_p(z)$$

is a holomorphic function in the closed polyhedron $\bar{\Delta}$. Relation (3.34) implies that

$$\begin{gathered} |h(z^0)| = 1, \\ |h(z)| < 1 \quad \text{for} \quad z \in \bar{\Delta} \backslash \{z^0\}. \end{gathered}$$

Hence it follows that the point $z^0 \in B(\Delta)$ and, since z^0 is an arbitrary point of the skeleton $S^*(\Delta)$, we have

$$S^*(\Delta) \subset B(\Delta) \subset S(\Delta). \tag{3.35}$$

From relations (3.32) and (3.35) it follows that $S^*(\Delta) = B(\Delta)$.

Now we shall show that $S(\Delta) = S^*(\Delta)$. For this purpose we consider the polyhedra $\Delta_\nu = \{|Z_j(z)| < 1 - \epsilon_\nu,\ j = 1, \cdots, N,\ z \in D\}$, where $\nu = 1, 2, \cdots, 1 > \epsilon_\nu > \epsilon_{\nu+1}$, $\lim_{\nu\to\infty} \epsilon_\nu = 0$. Then $\Delta_\nu \Subset \Delta_{\nu+1} \Subset \Delta$, $\lim_{\nu\to\infty} \Delta_\nu = \Delta$, $\lim_{\nu\to\infty} S^*(\Delta_\nu) = S^*(\Delta)$. In view of what has been proved above $S^*(\Delta_\nu) = B(\Delta_\nu)$. On the other hand, it follows from Theorem 15.5 to be proved below that $\lim_{\nu\to\infty} B(\Delta_\nu) \supset S(\Delta)$. Thus we find that

$$S^*(\Delta) \supset S(\Delta). \tag{3.36}$$

Relations (3.35) and (3.36) imply that $S^*(\Delta) = S(\Delta)$. Thus Theorem 15.4 is completely proved.

3. Some properties of the Šilov and Bergman boundaries.

THEOREM 15.5. *Let domains* $D_\nu \subset C^n$, $\nu \longrightarrow \infty$, *approximate from inside a bounded domain* $D \subset C^n$, *while* $D_\nu \subset D_{\nu+1} \Subset D$. *Then*

$$\lim_{\nu \to \infty} B(D_\nu) \supset S(D). \tag{3.37}$$

PROOF. Let $f(z)$ be a function holomorphic in the domain D and continuous in the closed domain $\bar{D}$; it is holomorphic in all closed domains $\bar{D}_\nu$. Take a number ν_ϵ so large that for $\nu > \nu_\epsilon$:

1) if $|f(z)| \leq M$ for $z \in \lim_{\nu \to \infty} B(D_\nu)$, then $|f(z)| \leq M + \epsilon/2$ for $z \in B(D_\nu)$ and accordingly also for $z \in \partial D_\nu$;

2) if $|f(z)| \leq M + \epsilon/2$ for $z \in \partial D_\nu$, then $|f(z)| \leq M + \epsilon$ for $z \in \partial D$.

Here $\epsilon > 0$ is a certain preassigned natural number. Such a choice of the number ν_ϵ is possible, since the domain D is bounded and the function $f(z)$ is holomorphic in the closed domain $\bar{D}$.

Hence it follows that if $|f(z)| \leq M$ for $z \in B(D_\nu)$ with $\nu > \nu_\epsilon$, then $|f(z)| \leq M + \epsilon$ for $z \in \partial D$. Since the number $\epsilon > 0$ is arbitrary, this can hold only in the case when the function $|f(z)|$ attains its largest value in the closed domain $\bar{D}$ on the subset $\lim_{\nu \to \infty} B(D_\nu)$ of the boundary ∂D. At the same time it has been established that the conclusion relation (3.37) holds. Theorems 15.5 is proved.

THEOREM 15.6. *Let domains* $D_\nu \subset C^n$, $\nu \longrightarrow \infty$, *approximate from inside a bounded domain* $D \subset C^n$, *while* $D_\nu \subset D_{\nu+1} \subset D$. *Then*

$$\lim_{\nu \to \infty} S(D_\nu) \supset S(D).$$

The proof of Theorem 15.6 is analogous to that of Theorem 15.5.

The following theorem is a strengthening of Theorem 15.3.

THEOREM 15.7. *Let* $\phi(z)$ *be a plurisubharmonic function in a domain* $B \subset C^n$, *and let the domain* $D = \{\phi(z) < 0\} \Subset B$. *If* $H(\phi) > 0$ *in some neighborhood of a point* $z^0 \in \partial D$ *for* $\sum_{k=1}^n |\alpha_k|^2 > 0$, *then the point* $z^0 \in B(D) \subset S(D)$.

PROOF. We shall carry out the proof for the function $\phi(z) \in \mathcal{C}^2$. We consider the function $\phi_1(z) = \phi(z) - \epsilon \sum_{k=1}^n |z_k - z_k^0|^2$, where a number $\epsilon > 0$ is to be selected so small that $H(\phi_1) > 0$ for $\sum_{k=1}^n |\alpha_k|^2 > 0$ in the hypersphere of a certain radius r with its center at the point z^0.

We take the function

$$u(z)=\begin{cases}\lambda\sum_{k=1}^{n}|z_k-z_k^0|^2, & \text{when } \sum_{k=1}^{n}|z_k-z_k^0|^2<\frac{1}{4}r^2,\\ \mu, & \text{when } \sum_{k=1}^{n}|z_k-z_k^0|^2>r^2.\end{cases}$$

Here the numbers λ, $\mu>0$, while $\lambda\leq\epsilon$. The value of the function $u(z)$ in the spherical annulus

$$\frac{1}{4}r^2\leqslant\sum_{k=1}^{n}|z_k-z_k^0|^2\leqslant r^2 \tag{3.38}$$

is so selected that in the whole space C^n the function $u(z)$ is positive, belongs to the class $\mathcal{C}^2$, and is such that

$$H(u)\leqslant H\left(\varepsilon\sum_{k=1}^{n}|z_k-z_k^0|^2\right).$$

The realization of the last inequality in the spherical annulus (3.38) is ensured by taking the number r sufficiently small, in view of the fact that the function $u(z)$ belongs to the class $\mathcal{C}^2$. We shall not go into the actual construction of the function $u(z)$. 1)

Then the function $\phi^*(z)=\phi(z)-u(z)$ will belong to the class $\mathcal{C}^2$ and turns out to be plurisubharmonic in the domain B. In addition $\phi^*(z^0)=0$ and

$$\phi^*(z)<0 \quad\text{for}\quad z\in\bar{D}\setminus\{z^0\}.$$

Starting from the function $\phi^*(z)$ we construct, by arguing in the same way as for the proof of Theorem 15.3, a holomorphic function whose modulus takes on its largest value in a preassigned arbitrary neighborhood of the point z^0. Hence it follows that the point $z^0\in S(D)$. Theorem 15.7 is proved

COROLLARY. *If the domain D and the function $\phi(z)$ satisfy the conditions of Theorem* 15.7, *then* $\partial D\cap\{H(\phi)>0\}\subset B(D)\subset S(D)$.

THEOREM 15.8. *Let a function $\phi(z)\in\mathcal{C}^2$ be plurisubharmonic in a domain $B\subset C^n$ and let the domain $D=\{\phi(z)<0\}\Subset B$. If $H(\phi)=0$ in some neighborhood of a point $z^0\in\partial D$, then $z^0\notin S(D)$.*

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1) See the similar constructions, for example, in the books of Bochner-Martin [1], (I), p. 230 ff. and of G. de Rham, *Variétés différentiables*, Actualités Sci. Ind. No 1222, Hermann, Paris, 1955; Russian transl., IL, Moscow, 1956.

PROOF. In the case under consideration $\phi(z)$ is a plurisubharmonic function in some neighborhood of the point z^0. In that neighborhood we may set $\phi(z) = \ln|g(z)|$, where $g(z)$ is a holomorphic function distinct from zero. If a number $r > 0$ is sufficiently small, then through each point of the set $\partial D \cap \{\Sigma_{k=1}^n |z_k - z_k^0|^2 < r^2\}$ there will pass the analytic surface $\Gamma = \{g(z) = e^{i\theta}\}$. The boundary of this surface will consist of the points of the set $\{\Sigma_{k=1}^n |z_k - z_k^0|^2 = r^2\} \cap \partial D$.

The analytic surface Γ is the limit of the analytic surfaces

$$\Gamma_\nu = \left[\{g(z) = \rho_\nu e^{i\theta}\} \cap \left\{\sum_{k=1}^n |z_k - z_k^0|^2 = r^2\right\}\right] \subset D,$$

where $\nu = 1, 2, \cdots$, $\lim_{\nu\to\infty}\rho_\nu = 1$. If the function $f(z)$ is holomorphic in the domain D (namely, also on the surface Γ_ν) and continuous in the closed domain $\bar{D}$, then its trace $f(z)|_\Gamma$ is the uniformly attained limit (as $\nu \longrightarrow \infty$) of the holomorphic traces $f(z)|_{\Gamma_\nu}$. Therefore the maximum principle holds for such functions on the surface Γ. If the modulus of such a function does not exceed the number M at the points $z \in \partial D \cap \{\Sigma_{k=1}^n |z_k - z_k^0|^2 = r^2\}$, then the same is also true at the points $z \in \partial D \cap \{\Sigma_{k=1}^n |z_k - z_k^0|^2 \leq r^2\}$. Hence it obviously follows that the point $z^0 \notin S(D)$. The theorem is proved.

The last two theorems provide sufficient visualization of the Šilov boundary of the domain of holomorphy.

On the basis of Theorem 15.8 one can show that the Šilov boundary of an analytic polyhedron of general form always makes up a part of the skeleton.

4. **Remark.** In the papers of Aĭzenberg [6], (I) and de Leeuw [1], (I), dealing with the integral representations of functions in n-circular domains, they investigated the boundaries of such domains in the sense of Šilov relative to polynomials. They showed that the Šilov boundaries of the complete n-circular domain D relative to polynomials coincide with their boundaries in the sense of Šilov relative to all the collection of functions holomorphic in the domain D and continuous in the closed domain $\bar{D}$.

§16. RELATIVE ANALYTIC CONVEXITY. APPLICATIONS TO THE THEORY OF APPROXIMATION

1. A domain analytically convex relative to its embracing domain.

DEFINITION. A domain of holomorphy $D \subset C^n$ is said to be *analytically convex relative to a domain* of holomorphy $D^* \subset C^n$ if the domain $D \subset D^*$ and

there exists a sequence of domains $D_\nu = \{\phi_\nu(z) < 0\}$ with the following properties:

1) $D_\nu \subset D_{\nu+1} \subset D$;

2) $\lim D_\nu = D$ as $\nu \longrightarrow \infty$;

3) all the functions $\phi_\nu(z)$ are plurisubharmonic in the domain D^*.

The concepts of analytic convexity and holomorphic convexity of one domain relative to another turn out to be equivalent. We have the

THEOREM 16.1 (Bremermann [4]). *Let D and D^* be domains of holomorphy of the space C^n. Then $D \Re D^*$ (i.e., the domain D is a Runge domain relative to the domain D^*) if and only if the domain D is analytically convex relative to the domain D^*.*

PROOF. I. First of all we establish the necessity of the condition. If $D \Re D^*$, then by Theorem 2.1 (see §3.3, Chapter I) the domain D is holomorphically convex relative to the domain D^*. In the course of the proof of Theorem 2.1 (see §2.2, Chapter I) it was shown that in such a case there exists a sequence of Weil analytic polyhedra D_ν which are connected components of the open sets

$$E_\nu = \{ | f_j^{(\nu)}(z) | < 1, \quad j = 1, \ldots, N_\nu \}, \qquad \nu = 1, 2, \ldots,$$

with the following properties:

1) $D_\nu \subset D_{\nu+1} \subset D$;

2) $\lim_{\nu \to \infty} D_\nu = D$;

3) all the functions $f_j^{(\nu)}(z)$ are holomorphic in the domain D^*.

Then the function

$$\varphi_\nu(z) = \sup \{ \ln | f_1^{(\nu)}(z) |, \ldots, \ln | f_{N_\nu}^{(\nu)}(z) | \}$$

is plurisubharmonic in the domain D^* by proposition 3) of Theorem 13.2 and $E_\nu = \{\phi_\nu(z) < 0\}$. Therefore our assertion is proved.

II. We shall now show that the condition is sufficient.

LEMMA 1. *If $D = \lim_{\nu \to \infty} D_\nu$, $D_\nu \subset D_{\nu+1} \subset D$ and $D_\nu \Re D^*$ for all values of the quantities ν, then $D \Re D^*$.*

PROOF. The relations $D_\nu \Re D^*$ imply that the domains D_ν are holomorphically convex relative to the domain D^*. Since the sequence $\{D_\nu\}$ approximates the domain D from inside, it follows that this domain is holomorphically convex relative to the domain D^* and accordingly (by Theorem 2.1) $D \Re D^*$.

LEMMA 2. *If all connected components of an open set $G \subset C^n$ are domains of holomorphy and for any function $f(z) \in \mathfrak{O}_G$*

$$\max_{z \in S} |f(z)| \leqslant \max_{z \in T} |f(z)|, \tag{3.39}$$

where $S \Subset G$, $T \Subset G$ are certain sets, then for any function $\phi(z)$ plurisubharmonic on the set G we also have

$$\max_{z \in S} \varphi(z) \leqslant \max_{z \in T} \varphi(z). \tag{3.40}$$

PROOF. First consider the case when $\phi(z) \in \mathcal{C}$. Then by Theorem 13.10, for every number $\epsilon > 0$, one can find, functions $f_1(z), \cdots, f_k(z) \in \mathfrak{O}_G$ and rational numbers $c_1, \cdots, c_k$, such that for $z \in G$

$$\varphi(z) \leqslant \sup [c_1 \ln |f_1(z)|, \ \ldots, \ c_k \ln |f_k(z)|] \leqslant \varphi(z) + \varepsilon. \tag{3.41}$$

From inequality (3.39) it follows that

$$\max_{z \in S} \sup [c_1 \ln |f_1(z)|, \ \ldots, \ c_k \ln |f_k(z)|] \leqslant \max_{z \in T} \sup [c_1 \ln |f_1(z)|, \ \ldots, \ c_k \ln |f_k(z)|].$$

Therefore from (3.41) we have also that

$$\max_{z \in S} \varphi(z) \leqslant \max_{z \in T} \varphi(z) + \varepsilon.$$

This last inequality holds for all numbers $\epsilon > 0$. Consequently inequality (3.40) is valid for the plurisubharmonic functions $\phi(z) \in \mathcal{C}$.

The assertion of Lemma 2 in the general case now follows from Theorem 13.9_1.

LEMMA 3. *If a function $\phi(z)$ is plurisubharmonic in a domain D^*, then for the domain $D = \{\phi(z) < 0, \ z \in D^*\}$ we have the relation: $D \Re D^*$.*

PROOF. Let a domain $D_0 \Subset D$ and $\rho = \sup_{z \in D_0} \phi(z)$; evidently $\rho < 0$. Consider the open set $D_\rho = \{\phi(z) < \rho, \ z \in D^*\}$; because of the semicontinuity of the function $\phi(z)$ we have

$$D_0 \subset D_\rho \Subset D.$$

Let a point $\zeta \in D \backslash \bar{D}_\rho$. The assumption that for all functions $f(z) \in \mathfrak{O}_{D^*}$,

$$|f(\zeta)| \leqslant \sup_{z \in D_0} |f(z)|$$

is found, in view of Lemma 2, to contradict the fact that for the plurisubharmonic

function $\phi(z)$

$$\varphi(\zeta) > \rho = \sup_{z \in D_0} \varphi(z)$$

(for the application of Lemma 2 one must put: $G = D^*$, $D_0 = S$, $T = \{\zeta\}$). Consequently there exists a function $f(z) \in \mathfrak{O}_{D^*}$ for which

$$|f(\zeta)| > \sup_{z \in D_0} |f(z)|.$$

This means that the domain D is holomorphically convex relative to the domain D^*. Lemma 3 is proved.

From Lemmas 3 and 1 there follows the sufficiency of the condition stated in Theorem 16.1, which completes the second part of the proof.

REMARK. For some applications the following formulation of Lemma 3 will be more convenient:

If $\phi(z)$ is a plurisubharmonic function in a domain of holomorphy $D \subset C^n$, then all connected components of the open set $\{\phi(z) < M,\ z \in D\}$, where M is a certain real number, are Runge domains relative to the domain D.

Hence there follows from Theorem 3.4 the

COROLLARY. *If $D \subset C^n$ is a domain of holomorphy, then all connected components of the open set $\{d_D(z)\,|\chi(z)| > \mu\}$, where μ is a certain positive number, are Runge domains relative to the domain D. Here the function $\chi(z)$ is holomorphic and nonvanishing in the domain D.*

Indeed, by Theorem 13.4 the function $-\ln d_D(z) - \ln|\chi(z)|$ is plurisubharmonic in the domain D. By applying the remark appended to Lemma 3 to the open set $\{-\ln d_D(z) - \ln|\chi(z)| < M\}$, where $M = \ln \mu$, we obtain our assertion.

2. The use of relative analytic convexity in the theory of approximation.

THEOREM 16.2 [1)] (Bremermann [4]). *Every convex domain $D \subset C^n$ is a Runge domain of the first kind.*

The proof of this theorem is carried out in the following way:

The domain D is convex if and only if the quantity $-\ln d_D(z)$ is a convex function for $z \in D$ (see the remark to Theorem 13.5). This function may be replaced in any domain $D_0 \Subset D$ with any desired accuracy by a function convex in the whole space C^n. Using the above-mentioned fact we construct a sequence of

1) This theorem also follows from the more general result of Behnke-Stein on the star-shaped domains of holomorphy (see §3.2, Chapter I).

functions $\phi_\nu(z)$, $\nu = 1, 2, \cdots$, convex in the whole space C^n, such that

$$D = \lim_{\nu\to\infty} \{\varphi_\nu(z) - M_\nu < 0\}, \tag{3.42}$$

where the numbers $M_\nu \longrightarrow \infty$ as $\nu \longrightarrow \infty$. Every convex function is a plurisubharmonic function (see Corollary 2 of Theorem 13.1). Therefore relation (3.42) and the remark to Lemma 3 imply that $D \subset C^n$. Thus every function $f(z) \in \mathfrak{O}_D$ can be uniformly approximated inside the domain D by entire functiбns; the latter may evidently be uniformly approximated in the domain D by polynomials. Hence our assertion follows.

As is well known, in order that the tubular domain be a domain of holomorphy, it is necessary and sufficient that it should be a convex domain (see §13.7 of the present chapter).

Therefore from Theorem 16.2 we have the following

COROLLARY. *The tubular domain of holomorphy is a Runge domain of the first kind.*

Hence one can derive the following conclusion:

Every function holomorphic in some tubular domain may be uniformly approximated therein by polynomials.

In many cases, in particular for the study of Šilov boundaries, the following theorem is useful:

THEOREM 16.3. *Consider the domain $D = \{\phi(z) < 0\}$, where $\phi(z)$ is a plurisubharmonic function in some domain $\widetilde{D} \Supset D$. Then for every function $f \in \mathfrak{O}_D$, domain $D_0 \Subset D$ and number $\epsilon > 0$ one can find a function $f^*(z) \in \mathfrak{O}_{\overline{D}}$, such that*

$$|f(z) - f^*(z)| < \varepsilon \quad \text{for} \quad z \in D_0.$$

Here $\mathfrak{O}_{\overline{D}}$ is the ring of functions holomorphic in the domain D and continuous in the closed domain $\overline{D}$.

In other words, *the functions constituting the normed ring $\mathfrak{O}_{\overline{D}}$ (with the norm equal to* $\max_{z \in \overline{D}} |f(z)|$, *where $f \in \mathfrak{O}_{\overline{D}}$) are dense in the sense of the topology induced by the uniform convergence in the domain D.*

PROOF. Indeed, $\mathfrak{O}_{\widetilde{D}} \subset \mathfrak{O}_{\overline{D}}$, while $D\mathfrak{R}\widetilde{D}$ in view of Theorem 16.1. Hence our assertion follows.

3. Approximation of functions and the Šilov boundary.

THEOREM 16.4.[1] *Let $D \subset C^n$ be a bounded open set, and let $\mathfrak{S}$ and $\mathfrak{S}_1$ be two collections of (complex or real) upper semicontinuous functions on the closure $\bar{D}$, while $\mathfrak{S} \supset \mathfrak{S}_1$ and the collection $\mathfrak{S}_1$ is dense in the collection $\mathfrak{S}$ in the sense of the topology of uniform convergence on this closure. Then*

$$S_{\mathfrak{S}}(D) = S_{\mathfrak{S}_1}(D). \tag{3.43}$$

From the existence and the uniqueness of the Šilov boundary on one side of the equality (3.43) *it follows that the Šilov boundary on the other side of the equality* (3.43) *exists and is defined uniquely.*

PROOF (carried out for the case of complex functions). I. Suppose it is known that there exists a Šilov boundary $S_{\mathfrak{S}_1}(D)$. Take an arbitrary function $\phi(z) \in \mathfrak{S}$. By our conditions there exists a sequence of functions $\phi_\nu(z) \in \mathfrak{S}_1$, $\nu = 1, 2, \cdots$, converging uniformly on the compactum $\bar{D}$ to the function $\phi(z)$. Let

$$\max |\phi_\nu(\bar{D})| = |\phi_\nu(z_\nu)|,$$

where $z_\nu \in S_{\mathfrak{S}_1}(D)$. Because of the compactness of the set $S_{\mathfrak{S}_1}(D)$, there exists the limit point $z_0 \in S_{\mathfrak{S}_1}(D)$ of the sequence z_ν, $\nu = 1, 2, \cdots$. Evidently $\max |\phi(\bar{D})| = |\phi(z_0)|$. Thus the set $S_{\mathfrak{S}_1}(D)$ is determining for the collection $\mathfrak{S}$ and, since $\mathfrak{S}_1 \subset \mathfrak{S}$, it is the minimum determining set for this collection.

We shall now show that it is defined uniquely. Let S be another minimum determining set for the collection $\mathfrak{S}$. Since $\mathfrak{S}_1 \subset \mathfrak{S}$, the set S is determining also for the collection $\mathfrak{S}_1$. Then there exists the minimum determining set $S^* \subset S$ of the collection $\mathfrak{S}_1$ (which is proved in the same way as for the first part of Theorem 15.1). From the uniqueness of $S_{\mathfrak{S}_1}(D)$, as the minimum determining set of the collection $\mathfrak{S}_1$, it follows that $S_{\mathfrak{S}_1}(D) = S^* \subset S$. But $S_{\mathfrak{S}_1}(D)$ and S are the determining sets of the collection $\mathfrak{S}$. Therefore $S_{\mathfrak{S}_1}(D) \subset S$ implies that $S_{\mathfrak{S}_1}(D) = S$. Thus it is proved that the set $S_{\mathfrak{S}}(D)$ exists and is defined uniquely and that $S_{\mathfrak{S}}(D) = S_{\mathfrak{S}_1}(D)$.

II. Suppose it is known that there exists a Šilov boundary $S_{\mathfrak{S}}(D)$. Since $\mathfrak{S}_1 \subset \mathfrak{S}$, the set $S_{\mathfrak{S}}(D)$ is determining for the collection $\mathfrak{S}_1$. Then there exists the

1) This theorem as well as the corollaries and examples related to it belong to L. A. Aĭzenberg.

minimum determining set $S \subset S_{\mathfrak{S}}(D)$ of this collection (which is again proved in the same way as for the first part of Theorem 15.1). Hence it follows from the first part of the present proof that $S = S_{\mathfrak{S}}(D)$ and consequently that $S_{\mathfrak{S}}(D)$ is the minimum determining set of the collection $\mathfrak{S}_1$.

We shall show that it is defined uniquely. Let S^* be another minimum determining set of the collection $\mathfrak{S}_1$. Then by the first part of the proof one can conclude that S^* is the minimum determining set for the collection $\mathfrak{S}$; hence in view of the uniqueness of the Šilov boundary $S_{\mathfrak{S}}(D)$ for the collection $\mathfrak{S}$ it is found that $S_{\mathfrak{S}}(D) = S^*$. Thus it is proved that the set $S_{\mathfrak{S}_1}(D)$ exists and is defined uniquely and that $S_{\mathfrak{S}_1}(D) = S_{\mathfrak{S}}(D)$.

COROLLARY 1. *If domains* D_0, D, $D^* \subset C^n$, $D_0 \Subset D \subset D^*$ *and* $D \Re D^*$, *then* $S_{\mathfrak{H}_D}(D_0) = S_{\mathfrak{H}_{D^*}}(D_0)$.

COROLLARY 2. *If every function holomorphic on a bounded open set* $D \subset C^n$ *and continuous on its closure* $\bar{D}$ *can be uniformly approximated on the compactum* $\bar{D}$ *by holomorphic functions (polynomials) on* $\bar{D}$, *then* $S(D) = B(D)$ *(respectively,* $S(D) = P(D)$*).*

COROLLARY 3. *If* $D \subset C^n$ *is a bounded open set and* $B(D) \neq S(D)$ *(respectively,* $P(D) \neq S(D)$*), then there exist functions, holomorphic on the set* D *and continuous on its closure, which cannot be uniformly approximated on the compactum* $\bar{D}$ *by holomorphic functions (polynomials) on* $\bar{D}$.

COROLLARY 4. *If a bounded domain* $D \subset C^n$ *has an exterior holomorphy hull* $N(D) \subset C^n$ *and the set* $S(D) \backslash S(N(D))$ *is not empty, then there exist functions, holomorphic in the domain* D *and continuous in the closed domain* $\bar{D}$, *which cannot be uniformly approximated on the compactum* $\bar{D}$ *by holomorphic functions on* $\bar{D}$.

EXAMPLES. 1. Consider the domain $D = \{0 < |w| < 1,\ |z| < |w|^{-\ln|w|}\}$. The functions $f_{\mu\nu} = w^{-\nu} z^{\mu}$, $\mu, \nu = 1, 2, \cdots$, are holomorphic in the domain D and continuous in the closed domain $\bar{D}$. The modulus of the function $f_{\mu\nu}$ attains its maximum in the closed domain $\bar{D}$ at the points (w, z), where $|w| = \exp(-\nu/\mu)$ and $|z| = |w|^{-\ln|w|}$. Hence on the basis of Theorem 16.4 one can see that $S(D) = \{|w| \leq 1,\ |z| = R(w)\}$, where $R(w) = |w|^{-\ln|w|}$ for $0 < |w| \leq 1$ and $R(0) = 0$.

The bicircular domain D has the complementary holomorphy hull $N(D) = \{|w| < 1,\ |z| < 1\}$. Hence it is easy to see that the Bergman boundary

$B(D) = \{|w| = 1, |z| = 1\}$. Thus $S(D) \neq B(D)$ in the present case. We have made mention of this fact in §15.1. By Corollary 3 there exist functions, holomorphic in the domain D and continuous on its closure $\bar{D}$, which cannot be uniformly approximated on the compactum $\bar{D}$ by holomorphic functions on $\bar{D}$. The functions $f_{\mu\nu}$ considered above serve as an example of such functions.

2. The fact that Theorem 16.4 is irreversible may be seen from the following example. Consider the domain $D = \{|w| < 1, |z| < |w|\} \subset C^2$ (D is a domain of holomorphy; it has the complementary holomorphy hull $N(D) = E = \{|w| < 1, |z| < 1\}$, see §5.4, Chapter I). In this case $S(D) = B(D) = \{|w| = |z| = 1\}$. Take the function $w^{-1}z^2 \in \mathfrak{H}_D$. It is continuous on the compactum $\bar{D}$, but it cannot be uniformly approximated on this compactum $\bar{D}$ by holomorphic functions on $\bar{D}$. Indeed, if such an approximation is possible on the skeleton $\{|w| = |z| = 1\}$ of the bicylinder E, then it would follow that the same is true also in the bicylinder E itself (which is readily seen by means of Cauchy's integral formula). This last is evidently impossible since the origin of coordinates is the singular point of the function $w^{-1}z^2$.

Thus $\mathfrak{S} \supset \mathfrak{S}_1$ and $S_{\mathfrak{S}}(D) = S_{\mathfrak{S}_1}(D)$ (in our example $\mathfrak{S}$ is the collection of functions holomorphic in the domain D and continuous on its closure $\bar{D}$, $\mathfrak{S}_1$ is the collection of functions holomorphic in the closed domain $\bar{D}$, $S_{\mathfrak{S}}(D) = S(D)$ and $S_{\mathfrak{S}_1}(D) = B(D)$) do not imply that the collection $\mathfrak{S}_1$ is dense in the collection $\mathfrak{S}$ in the sense of the topology of uniform convergence in the closed domain $\bar{D}$.

THEOREM 16.5. *Let the domain $D = \{\phi(z) < 0\}$, where $\phi(z)$ is a continuous plurisubharmonic function in some domain $\tilde{D} \Supset D$. Then $S_{\mathfrak{S}}(D) = B(D)$.*

The proof of this theorem is carried out in the following way:

It follows from Theorems 16.4 and 13.10 that the Šilov boundary $S_{\mathfrak{H}_{\tilde{D}}}(D) = \tilde{S}(D)$ coincides with the Šilov boundary of the domain D relative to functions continuous and plurisubharmonic in the domain $\tilde{D}$.

In view of Theorem 13.9_1 an arbitrary plurisubharmonic function $\psi(z)$ in the domain $\tilde{D}$ may be approximated by a monotonically decreasing sequence of continuous plurisubharmonic functions.

Hence it follows that if such a function $\psi(z) < M$ for $z \in \tilde{S}(D)$, then $\psi(z) < M$ also for $z \in \bar{D}$.

On the other hand, $\widetilde{S}(D)$ is the smallest subset of the closed domain D that has the above-mentioned property. Indeed, the collection $\mathfrak{S}$ of functions plurisubharmonic in the domain $\widetilde{D}$ contains as a part the collection of moduli of functions holomorphic in the domain $\widetilde{D}$. For the latter set $\widetilde{S}(D)$ is minimal in the above sense; consequently it is minimal also for the class of plurisubharmonic functions under consideration. Thus $S_{\mathfrak{S}}(D) = \widetilde{S}(D)$.

Let a function $f(z) \in \mathfrak{O}_{\overline{D}}$. Then one can select a number $\epsilon > 0$ so that $f(z) \in \mathfrak{O}_{D_1}$, where D_1 is a connected component of the set $\{\phi(z) < \epsilon\}$ which contains the domain D, while $D \Subset D_1 \Subset \widetilde{D}$. But from Theorem 16.1 we have $D_1 \Re \widetilde{D}$. Therefore the function $f(z)$ can be uniformly approximated on the compactum $\overline{D}$ by functions holomorphic in the domain $\widetilde{D}$. Hence $B(D) = \widetilde{S}(D)$ by Theorem 16.4 and, accordingly, $S_{\mathfrak{S}}(D) = B(D)$.

CHAPTER IV

HOLOMORPHIC EXTENSION OF DOMAINS

§17. GENERAL METHODS OF HOLOMORPHIC EXTENSION OF DOMAINS

1. Survey of results.

DEFINITION (holomorphic extension of domains). Let D and D^* be domains over the space P^n, $D < D^*$. If all functions holomorphic in the domain D are analytically continuable to the domain D^*, then this latter domain is called a *holomorphic extension* of the domain D.

A domain D not allowing holomorphic extension is a domain of holomorphy. The holomorphy hull $H(D)$ of a domain D serves as its maximum holomorphic extension.

The concept of holomorphic extension can be generalized without any change to a domain of an arbitrary complex space.

For many classes of domains the problem of constructing their holomorphy hull (more properly, the problem of giving a geometrical description of such hulls) is always difficult and at present is still far from solution. Therefore it is meaningful to investigate the methods of constructing domains which are (generally speaking, not maximum) holomorphic extensions of a given domain.

For example, in the course of the proof of the dispersion relations in quantum field theory one considers the analytic continuation of scattering matrix elements. First the "primitive" domains are found, in which it is comparatively easy to establish the holomorphy of the above elements. Then there arises the problem of the maximum holomorphic extension of the primitive domains. So far no one has succeeded in finding a solution of this problem in a general form, but in several cases it turns out to be possible to determine an essential holomorphic extension of the primitive domains of holomorphy. This allows us to prove the validity of the dispersion relations for some finite intervals for the variables of

momentum transfer. [1)]

An analogous situation occurs in the study of the analytic properties of the vacuum expectation values of field operators (the Wightman functions). [2)]

Beginning with the holomorphic extension of a domain or an open set consisting of a certain series of domains, one can consider their K-extensions. If a domain D or, in the more general case, a set D makes up a part of the intersection $\mathfrak{K}$ of domains of holomorphy for some collection K of holomorphic functions, then as a result of such K-extensions a certain part of the set $\mathfrak{K} \backslash D$ is adjoined to D.

Many theorems proved in the preceding volume afford ways of forming holomorphic extension of corresponding domains.

First of all we include Hartogs' Theorems 6.2, (I)–6.4, (I) (see §6, Chapter I, (I)) on the possibility of the analytic continuation of a function, holomorphic on a definite part of a polycylindrical domain, to the whole of that polycylindrical domain. These theorems imply that some domain $G \subset C^n$, containing the part of such polycylindrical domains that is indicated in these theorems, can be holomorphically extended by means of complete adjoining.

Theorem 21.2 (I) (see §21, Chapter IV, (I)) establishes that any function holomorphic in a neighborhood of the boundary ∂D of a bounded domain $D \subset C^n$ (the boundary ∂D must be connected) is analytically continuable to the domain D. Therefore a domain $G \subset C^n$ containing the above neighborhood of the boundary ∂D can be holomorphically extended by adjoining to it the domain D.

For a tubular domain holomorphic extension is reduced to constructing its convex hull.

Some theorems which may be useful for holomorphic extension are due to S. Bochner and W. Martin. [3)]

2. Edge Theorem. We shall devote this subsection to the deduction of a theorem that is frequently used for the holomorphic extension of domains in the space C^2 of variables w and z.

THEOREM 17.1 (Edge Theorem; Kneser [1]). *Let hypersurfaces* $\{\Phi = 0\}$ *and*

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1) See Vladimirov [1]. Further literature is cited in this paper.

2) See A. S. Wightman, *Quantum field theory and analytic functions of several complex variables*, J. Indian Math. Soc. 24 (1960–1961), 625–677.

3) See Bochner-Martin [1], (I), Chapter IV, §2.

$\{\Psi = 0\}$ *where* $\Phi, \Psi \in C^2$, *intersect in a surface* S *and at every point of the surface* S *let these hypersurfaces have different tangent hyperplanes. Then if there exists a domain of holomorphy* $D \subset C^2$ *such that:* 1) $S \subset \partial D$; 2) *every point of the surface* S *has a neighborhood* U *for which the intersection* $U^* = U \cap \{\max(\Phi, \Psi) > 0\} \subset D$, *then* S *is an analytic surface.*

REMARK. From this theorem it follows, for example, that there cannot exist a function which is holomorphic in three of the four parts into which the space C^2 is divided by the hypersurface $\{|w| = 1\}$, $\{|z| = 1\}$, and such that all the points of their intersection are singular points of the function.

PROOF. The condition that the hypersurfaces have different tangent hyperplanes at points of the surface S means that there we have

$$\operatorname{Rank}\begin{Vmatrix} \Phi'_w & \Phi'_z & \Phi'_{\bar w} & \Phi'_{\bar z} \\ \Psi'_w & \Psi'_z & \Psi'_{\bar w} & \Psi'_{\bar z} \end{Vmatrix} = 2. \tag{4.1}$$

We suppose that in our case

$$\Phi'_w \Psi'_z - \Psi'_w \Phi'_z \neq 0 \tag{4.2}$$

and show that this contradicts the assumption that the points of S belong to the boundary of the domain D. Let $P(a, b)$ be a certain point of the surface S. We consider the analytic surface

$$(V) \qquad \left.\begin{aligned} w &= a + a_1 t + {}^1/_2 a_2 t^2, \\ z &= b + b_1 t + {}^1/_2 b_2 t^2, \end{aligned}\right\} \tag{4.3}$$

given in parametric form. Here a_1, a_2, b_1, b_2 are constants to be determined below, and t is a complex parameter. The point P evidently corresponds to the value $t = 0$ of this parameter. We consider the values of the functions Φ and Ψ at points of this surface and substitute into these functions the quantities w, z, $\bar w$, $\bar z$ from relations (4.3). Denote by Φ^*, Ψ^* the expressions obtained from Φ and Ψ as a result of this substitution. Then it is evident that

$$\Phi^{*\prime}_t = \Phi'_w (a_1 + a_2 t) + \Phi'_z (b_1 + b_2 t),$$

$$\begin{aligned} \Phi^{*\prime\prime}_{t^2} = \Phi''_{w^2}(a_1 + a_2 t)^2 + 2\Phi''_{wz}(a_1 + a_2 t)(b_1 + b_2 t) \\ + \Phi''_{z^2}(b_1 + b_2 t)^2 + \Phi'_w a_2 + \Phi'_z b_2, \end{aligned}$$

$$\begin{aligned} \Phi^{*\prime\prime}_{t\bar t} = \Phi''_{w\bar w}(a_1 + a_2 t)\overline{(a_1 + a_2 t)} + \Phi''_{w\bar z}(a_1 + a_2 t)\overline{(b_1 + b_2 t)} \\ + \Phi''_{z\bar w}(b_1 + b_2 t)\overline{(a_1 + a_2 t)} + \Phi''_{z\bar z}(b_1 + b_2 t)\overline{(b_1 + b_2 t)}. \end{aligned}$$

The derivatives with respect to t and $\bar{t}$ of the function Ψ can be calculated analogously. We choose the quantities a_1, b_1 so that for $t = 0$ (i.e., at the point P) we shall have

$$\left.\begin{aligned}(\Phi_t^{*'})_{t=0} &= a_1(\Phi'_w)_P + b_1(\Phi'_z)_P = 1,\\ (\Psi_t^{*'})_{t=0} &= a_1(\Psi'_w)_P + b_1(\Psi'_z)_P = -1.\end{aligned}\right\} \tag{4.4}$$

Such a choice is possible in view of the condition (4.2). Also, the quantities a_2, b_2 are so chosen that we shall have

$$\begin{aligned}(\Phi^{*'}_{t\bar{t}})_{t=0} - (\Phi^{*''}_{t^2})_{t=0} = (\Phi''_{w^2})_P a_1^2 + 2(\Phi''_{wz})_P a_1 b_1 + (\Phi''_{z^2})_P b_1^2\\ + (\Phi'_w)_P a_2 + (\Phi'_z)_P b_2 - (\Phi''_{w\bar{w}})_P a_1\bar{a}_1 - (\Phi''_{w\bar{z}})_P a_1\bar{b}_1\\ - (\Phi''_{z\bar{w}})_P b_1\bar{a}_1 - (\Phi''_{z\bar{z}})_P b_1\bar{b}_1 = 1,\end{aligned}$$

$$(\Psi^{*''}_{t\bar{t}})_{t=0} - (\Psi^{*''}_{t^2})_{t=0} = 1 \tag{4.5}$$

(in detail, the latter equation is similar to the former). Such a choice is again possible in view of the condition (4.2). Then, if we put $t = u + iv$, we evidently have, for $t = 0$ (on account of equalities (4.4), (4.5) and their conjugate expressions):

$$\left.\begin{aligned}&(\Phi_u^{*'})_0 = 2,\ (\Psi_u^{*'})_0 = -2,\ (\Phi_v^{*'})_0 = 0,\ (\Psi_v^{*'})_0 = 0,\\ &(\Phi^{*''}_{v^2})_0 = -(\Phi^{*''}_{t^2})_0 + 2(\Phi^{*''}_{t\bar{t}})_0 - (\Phi^{*''}_{\bar{t}^2})_0\\ &\quad = [(\Phi^{*''}_{t\bar{t}})_0 - (\Phi^{*''}_{t^2})_0] + [(\Phi^{*''}_{t\bar{t}})_0 - (\Phi^{*''}_{\bar{t}^2})_0] = 2,\\ &(\Psi^{*''}_{v^2})_0 = 2\end{aligned}\right\} \tag{4.6}$$

(the last equality is obtained in the same way as the preceding one).

By relations (4.6), the expansions of the functions Φ^* and Ψ^* in the neighborhood of the point P will have the form:

$$\left.\begin{aligned}\Phi^* &= 2u + {}^1/_2(r_1u^2 + 2s_1uv + 2v^2) + \dots,\\ \Psi^* &= -2u + {}^1/_2(r_2u^2 + 2s_2uv + 2v^2) + \dots.\end{aligned}\right\} \tag{4.7}$$

These expansions show that in the neighborhood of the point P the quantity Φ^* or the quantity Ψ^* will be positive for $u \neq 0$, while for $u = 0$ but $v \neq 0$, both functions Φ^* and Ψ^* turn out to be larger than zero. Thus in the neighborhood U of the point P on the surface (4.3) at least one of the functions Φ, Ψ is greater than zero, and accordingly

$$V \cap U = V \cap U^*.$$

We further define two complex numbers p, q in such a way that

$$\operatorname{Re}\left[(\Phi'_w)_P\, p + (\Phi'_z)_P\, q\right] > 0, \quad \operatorname{Re}\left[(\Psi'_w)_P\, p + (\Psi'_z)_P\, q\right] > 0. \tag{4.8}$$

Evidently this can be done by equating the quantities in the square brackets to numbers with positive real parts and by solving the resulting equations for p and q. The latter is possible in view of the condition (4.2).

Next we consider the family of analytic surfaces:

$$\left.\begin{aligned} w &= a + \alpha p + a_1 t + {}^1\!/_2 a_2 t^2, \\ z &= b + \alpha q + b_1 t + {}^1\!/_2 b_2 t^2. \end{aligned}\right\} \tag{4.9}$$

It is easy to see that the family is regular in a neighborhood of the point P and the value $\alpha = 0$, for which we obtain the surface (4.3) already considered. Then, by Theorem 12.1, (I) (see Chapter II, (I)), on each surface of the family (4.9) for $|\alpha| < \delta$ (where δ is a small positive number) there must lie, in some neighborhood of the point P, points not belonging to the domain D.

On the other hand, we put $\alpha = s > 0$ and form the values of the functions Φ and Ψ for the surfaces (4.9) with this value of α. We denote by $\Phi(t, s)$ and $\Psi(t, s)$ the result of the substitution of (4.9) in the functions Φ and Ψ with $\alpha = s$. Then in view of the inequalities (4.8) we obtain

$$\left[\Phi'_s(t, s)\right]_{t,\, s=0} = (\Phi'_w)_P\, p + (\Phi'_z)_P\, q + (\Phi'_{\bar w})_P\, \bar p + (\Phi'_{\bar z})_P\, \bar q > 0,$$

$$\left[\Psi'_s(t, s)\right]_{t,\, s=0} = (\Psi'_w)_P\, p + (\Psi'_z)_P\, q + (\Psi'_{\bar w})_P\, \bar p + (\Psi'_{\bar z})_P\, \bar q > 0.$$

Hence, because of the continuity of the derivatives of the functions Φ and Ψ, it follows that for small values of the parameters s and t the quantities $\Phi(t, s)$ and $\Psi(t, s)$ are increasing functions of s. But for $s = 0$ and small $|t|$

$$\max\left[\Phi(t, 0), \Psi(t, 0)\right] > 0;$$

then for small $s > 0$ and $|t|$

$$\max\left[\Phi(t, s), \Psi(t, s)\right] > 0.$$

Therefore, by the assumptions of the theorem, the surface (4.9) within the limits of the neighborhood U lies inside the set U^* and, accordingly, in the domain D. Thus we have arrived at a contradiction and must reject the assumption expressed by inequality (4.2).

Thus, under the assumptions of our theorem, we have

$$\frac{\Phi'_w}{\Psi'_w} = \frac{\Phi'_z}{\Psi'_z} = \lambda. \tag{4.10}$$

We further remark that in our case $\lambda \neq \overline{\lambda}$ (since if $\lambda = \overline{\lambda}$, then the condition (4.1) could not be valid).

If the differentials dw, dz define a displacement in the tangent plane to the surface S, then one can evidently write

$$\Phi'_w\, dw + \Phi'_z\, dz + \Phi'_{\overline{w}}\, d\overline{w} + \Phi'_{\overline{z}}\, d\overline{z} = 0, \tag{4.11}$$

$$\Psi'_w\, dw + \Psi'_z\, dz + \Psi'_{\overline{w}}\, d\overline{w} + \Psi'_{\overline{z}}\, d\overline{z} = 0. \tag{4.12}$$

In view of relations (4.10), equality (4.12) can be rewritten as

$$\lambda\left(\Phi'_w\, dw + \Phi'_z\, dz\right) + \overline{\lambda}\left(\Phi'_{\overline{w}}\, d\overline{w} + \Phi'_{\overline{z}}\, d\overline{z}\right) = 0. \tag{4.13}$$

From relations (4.11) and (4.13) it follows (since $\lambda \neq \overline{\lambda}$) that on the surface S we have

$$\Phi'_w\, dw + \Phi'_z\, dz = 0; \qquad \Phi'_{\overline{w}}\, d\overline{w} + \Phi'_{\overline{z}}\, d\overline{z} = 0.$$

Hence, applying the usual rule for the differentiation of implicit functions, we find that $\partial w / \partial \overline{z} = 0$ on the surface S. Consequently the surface S is analytic. Theorem 17.1 is proved.

3. Bremermann's process of holomorphic extension.

THEOREM 17.2. [1] *Let:*

1) *D be a domain in the space C^n of complex variables $z_1, \cdots, z_n$;*

2) *S be a simply-connected domain lying, along with its boundary ∂S, on the intersection of the domain D with an analytic plane $z_k = b_k \lambda + z_k^0$, $k = 1, \cdots, n$, where z_k^0, b_k are complex constants and λ is a complex parameter;*

3) *S^* be the preimage of the domain S in the plane of the complex parameter λ;*

4) *$\chi(\lambda)$ be a holomorphic function at the points $\lambda \in S^*$, while its modulus $|\chi(\lambda)|$ is assumed to be continuous and distinct from zero for $\lambda \in \overline{S}^*$;*

1) This theorem, together with the corollaries obtained from it below, remains correct in a more general case, namely for domains in an infinite-dimensional Banach space, with boundary distances calculated in a more general metric. Cf. Bremermann [3], p. 23 and [1], (I).

5) $d_{a,D}(z)$ *be the radius of the largest disk with its center at a point* $z(z_1, \cdots, z_n) \in D$, *lying in the plane* $Z_k = a_k t + z_k$, $k = 1, \cdots, n$, *and consisting of points of the domain* D. *Here* t *is a complex parameter and* $|a_1|^2 + \cdots + |a_n|^2 = 1$.

Then if

$$|\chi(\lambda)|\, d_{a,D}(z^0 + \lambda b) \geqslant m > 0 \quad \textit{for} \quad \lambda \in \partial S^*, \tag{4.14}$$

where $z^0 + \lambda b$ *is the point with coordinates* $z_k^0 + \lambda b_k$, $k = 1, \cdots, n$, *every function* $f(z)$ *holomorphic in the domain* D *is analytically continuable to points of the set*

$$G_a = \{z_k = z_k^0 + \lambda b_k + \tau a_k,\ k = 1, \ldots, n\}, \tag{4.15}$$

where $\lambda \in \bar{S}^*$, $|\tau| < m\,|\chi(\lambda)|^{-1}$. *Here* τ *is a complex parameter.*

PROOF. We denote by E the section of the domain D by the three-complex-dimensional plane given in the parametric form

$$z_k = z_k^0 + \lambda b_k + \tau a_k + \rho c_k, \qquad k = 1, \ldots, n. \tag{4.16}$$

Here ρ (like λ and τ) is a complex parameter and $|c_1|^2 + \cdots + |c_n|^2 = 1$. Let E^* be the preimage of the set E in the space of the complex variables λ, τ, ρ. Then to every function $f(z)$ holomorphic in the domain D there corresponds a function $\tilde{f}(\lambda, \tau, \rho)$, obtained from it by the substitution (4.16) and holomorphic for $(\lambda, \tau, \rho) \in E^*$.

Because of the fourth assumption of the theorem and the condition (4.14) one can find for each number $\epsilon > 0$ a number $\delta > 0$ such that the set

$$H_{\varepsilon,\delta,c} = \{z_k = z_k^0 + \lambda b_k + \tau a_k + \rho c_k,\quad k = 1, \ldots, n,$$
$$|\tau| \leqslant (m - \varepsilon)|\chi(\lambda)|^{-1},\ \lambda \in \partial S^*,\ |\rho| \leqslant \delta\} \Subset D,$$

for an arbitrary vector $c(c_1, \cdots, c_n)$. Since the set $H_{\epsilon,\delta,c}$ is strictly contained in the set D, we have

$$|\tilde{f}(\lambda, \tau, \rho)| < M$$

for $(\lambda, \tau, \rho) \in H^*_{\epsilon,\delta,c}$. Here $H^*_{\epsilon,\delta,c}$ is the preimage of the set $H_{\epsilon,\delta,c}$ in the space of the parameters λ, τ, ρ, and $M = M(\epsilon, \delta, c, f)$ is some positive number.

We expand the function $\tilde{f}$ in a power series of the variables τ and ρ in a neighborhood of the point $\tau = 0$, $\rho = 0$. As a result we obtain

$$f(z^0 + \lambda b + \tau a + \rho c) = \sum_{\mu,\nu=0}^{\infty} \frac{1}{\mu!\,\nu!} \left. \frac{\partial^{\mu+\nu} f(z_0 + \lambda b + \tau a + \rho c)}{\partial \tau^\mu\, \partial \rho^\nu} \right|_{\tau=\rho=\nu} \tau^\mu \rho^\nu. \tag{4.17}$$

Hence, in view of Cauchy's inequalities for the coefficients of the power series, we have

$$\frac{1}{\mu!\,\nu!}\left|\left[\frac{\partial^{\mu+\nu} f(z^0+\lambda b+\tau a+\rho c)}{\partial\tau^{\mu}\,\partial\rho^{\nu}}\right]_{\tau=\rho=0}\right| \leqslant \frac{M}{[(m-\varepsilon)\,|\chi(\lambda)|^{-1}]^{\mu}\,\delta^{\nu}}$$

or

$$\frac{1}{\mu!\,\nu!}\left|\left[\frac{\partial^{\mu+\nu} f(z^0+\lambda b+\tau a+\rho c)}{\partial\tau^{\mu}\,\partial\rho^{\nu}}\right]_{\tau=\rho=0}\right| |\chi(\lambda)|^{-\mu} \leqslant \frac{M}{(m-\varepsilon)^{\mu}\,\delta^{\nu}} \tag{4.18}$$

for all $\mu, \nu = 0, 1, 2, \cdots$ and $\lambda \in \partial S^*$.

On the left-hand side of the inequality there is a function of λ, holomorphic on the domain S^* and continuous in the closed domain $\bar{S}^*$. Therefore inequality (4.18) holds not only for $\lambda \in \partial S^*$ but also for $\lambda \in \bar{S}^*$. Hence it follows that series (4.17) converges uniformly on the set $\bar{\mathfrak{H}}_{\epsilon,\delta,c}$, where

$$\mathfrak{H}_{\varepsilon,\delta,c} = \{z_k = z_k^0 + \lambda b_k + \tau a_k + \rho c_k,\quad k = 1, \ldots, n,$$
$$|\tau| < (m-\varepsilon)\,|\chi(\lambda)|^{-1},\ \lambda \in \bar{S}^*,\ |\rho| < \delta\},$$

and defines a continuation of the function $\tilde{f}$.

Assigning various directions to the vector c and then letting the quantity ϵ tend to zero, we thus continue the function $f(z)$ to the complete neighborhood of the point set G_a.

One and the same point of the set G_a can belong to several sets $\mathfrak{H}_{\epsilon,\delta,c}$. Therefore we must show that by such a continuation we do not obtain a many-valued function.

Take $c = c'$ on the one hand, and $c = c''$ on the other. If the vectors a, b, c', and c'' are linearly independent, then for all values of $(\lambda_1, \tau_1, \rho_1) \neq (\lambda_2, \tau_2, \rho_2)$,

$$z' = z^0 + \lambda_1 b + \tau_1 a + \rho_1 c' \neq z'' = z^0 + \lambda_2 b + \tau_2 a + \rho_2 c''.$$

In this case we never obtain two different values of the continued function $f(z)$ at one and the same point.

Assume that the vectors a, b, c', c'' are linearly dependent, i.e., $c'' = \alpha_1 a + \alpha_2 b + \alpha_3 c'$. Then it turns out that $z' = z''$ for

$$\lambda_1 = \lambda_2 + \rho_2\alpha_2,\quad \tau_1 = \tau_2 + \rho_2\alpha_1,\quad \rho_1 = \rho_2\alpha_3.$$

Let

$$f(z^0+\lambda b+\tau a+\rho c')=f_1(\lambda,\ \tau,\ \rho);$$
$$f(z^0+\lambda b+\tau a+\rho c'')=f_2(\lambda,\ \tau,\ \rho).$$

In some neighborhood of the point (0, 0, 0) in the space of the variables λ, τ, ρ (lying in the domain of the original values of the function f) these functional elements coincide and we have the relation

$$f_1(\lambda+\rho\alpha_2,\ \tau+\rho\alpha_1,\ \rho\alpha_3)\equiv f_2(\lambda,\ \tau,\ \rho).$$

Evidently this relation is preserved under direct analytic continuation.

Thus we have established the single-valuedness of the function $f(\lambda, \tau, \rho)$ obtained by this continuation. The domain G_a is simply-connected since the planar domain S is simply-connected. Hence we conclude that the function $f(z)$ defined in the domain G_a by our process of analytic continuation is single-valued. The theorem is proved.

THEOREM 17.3. *Let the conditions* 1)–4) *of the preceding theorem be satisfied, and let*

5) $d_D(z)$ *be the boundary distance of a point* $z(z_1, \cdots, z_n) \in D$ *to the boundary* ∂D *of the domain* D.

Then if

$$|\chi(\lambda)|\, d_D(z^0+\lambda b)\geqslant m>0 \quad \text{for } \lambda\in\partial S^*, \tag{4.14$_1$}$$

every function $f(z)$ *that is holomorphic in the domain* D *is analytically continuable to points of the set*

$$G=\left\{\left[\sum_{k=1}^{n}|z_k^0+\lambda b_k-z_k|^2\right]^{1/2}<m\,|\chi(\lambda)|^{-1}\right\}, \tag{4.15$_1$}$$

where $\lambda \in \bar{S}^*$.

PROOF. It follows from (4.14$_1$) that inequality (4.14) holds for all $\lambda \in \partial S^*$ and for any unit vector a. Therefore in view of Theorem 17.2 the function $f(z)$ is continuable to points of the set G_a. But $G = \bigcup_{(a)} G_a$. Hence our assertion follows because of the simple-connectedness of the domain G (which ensures the single-valuedness of the continued function). The theorem is proved.

By choosing a function $\chi(\lambda)$ in the domain S^* in different ways, we shall obtain different domains G. We have

THEOREM 17.4 (Bremermann [1], (I)). *Let domains* D *and* S^* *satisfy the conditions of the preceding theorem. Then the largest domain* G *corresponds to*

choosing as $\chi(\lambda)$ *the function*

$$\chi_0(\lambda) = \exp[h(\lambda) + ih^*(\lambda)], \tag{4.19}$$

where $h(\lambda)$ *is a harmonic function in the domain* S^* *satisfying the condition*

$$h(\lambda) = -\ln d_D(z_0 + \lambda b) \text{ for } \lambda \in \partial S^*; \tag{4.20}$$

$h^*(\lambda)$ *is the harmonic function in the domain* S^* *conjugate to the function* $h(\lambda)$.

PROOF. In view of Theorem 13.4 the function $d_D(z)$ is continuous; by assumption, the domain S^* is simply-connected. Therefore the Dirichlet problem for the condition (4.20) is solvable, and the function $h(\lambda) + ih^*(\lambda)$ is single-valued and accordingly holomorphic in the domain S^*. Evidently the function $\chi_0(\lambda)$ defined by formula (4.19) satisfies all the conditions of the preceding theorem, since

$$\inf d_D(z^0 + \lambda b)\,|\chi_0(\lambda)|^{-1} = 1.$$

We must show that

$$G(S, \chi_0) \supset G(S, \chi)$$

for any function $\chi(\lambda)$, satisfying the conditions of Theorem 17.3 and distinct from the function $\chi_0(\lambda)$.

Put

$$\chi(\lambda) = \exp[g(\lambda) + ig^*(\lambda)];$$

let

$$m = \inf_{s \in \partial S^*} d_D(z^0 + \lambda b)\,|\chi(\lambda)| \quad \text{for} \quad \lambda \in \partial S^*.$$

Then

$$\ln d_D(z^0 + \lambda b) + g(\lambda) \geqslant \ln m \quad \text{for} \quad \lambda \in \partial S^*,$$

and accordingly, because of equality (4.20), we have

$$-h(\lambda) \geq \ln m - g(\lambda) \text{ for } \lambda \in \partial S^*. \tag{4.21}$$

Hence, since $h(\lambda)$ and $g(\lambda)$ are harmonic functions in the domain S^*, it follows that inequality (4.21) also remains valid for the points $\lambda \in \bar{S}^*$. Therefore

$$|\chi_0(\lambda)|^{-1} \geqslant m\,|\chi(\lambda)|^{-1},$$

$$\left\{ \sum_{k=1}^{n} |z_k - z_k^0 - \lambda b_k|^2 < m^2 |\chi(\lambda)|^{-2} \right\}$$
$$\subseteq \left\{ \sum_{k=1}^{n} |z_k - z_k^0 - \lambda b_k|^2 < |\chi_0(\lambda)|^{-2} \right\}$$

for $\lambda \in S^*$. Hence the assertion of the theorem follows.

In what follows, in applying Theorem 17.3, we shall as a rule choose the function $\chi(\lambda)$ as indicated in Theorem 17.4.

Let G be any domain satisfying the conditions of Theorem 17.3 (but not necessarily Theorem 17.4). If the intersection $D \cap G$ is connected, then the values of the function $f(z)$ which are obtained in the domain G by our process of analytic continuation are determined uniquely.

In this case the domain $D \cup G$ is a holomorphic extension of the domain D; we shall call the passage from the domain D to the domain $D \cup G$ *Bremermann's process of holomorphic extension of the domain* D.

If the intersection $D \cap G$ is not connected, we may obtain a many-valued function by the continuation process. By considering the intersection of domains of holomorphy for all functions obtained here with respect to the connected components of the intersection $D \cap G$ which contains the domain S (see §8.5, Chapter II, (I)), we obtain a domain over the space C^n. The set $D \cup G$ will be the totality of the fundamental points of this domain. In the next subsection we will consider the process of constructing this domain for the more general case when the initial domain D is also multiple-sheeted.

4. **Bremermann's process of holomorphic extension for domains over the space C^n.** Let D be a domain over the space C^n, and assume that the sets S, $T \subset U \subset D$, where U is some single-sheeted subset of the domain D. It is supposed that the projection ηS (where η is the projection associating with points $P \in D$ their ground points $\underline{P}$) is a convex domain on some analytic plane $z_k = z_k^0 + \lambda b_k$, $k = 1, \cdots, n$, in the space C^n, while the projection $\eta T = \partial(\eta S)$ is its boundary.

We again consider the function $\chi(P) = \chi(\lambda)$, where $z^0 + \lambda b = \eta P$, and thus there corresponds to each point $P \in S \cup T$ a point λ in the closed domain $\overline{S}^*$ on the plane of the complex parameter λ. Let $m = \inf_{P \in T} d_D(P)\,|\chi(P)|$, where $d_D(P)$ is, as before, the radius of the largest single-sheeted ball with its center at the point P that belongs to the domain D. Then

$$d_D(P) \geq m\,|\chi(P)| \text{ for } P \in T.$$

We denote by $B(P, r) \subset D$ the single-sheeted ball of radius $r \leq d_D(P)$ with its center at the point $P \in D$. Then

$$\bigcup_{P\in T} B(P,\ d_D(P))\subset D,$$

$$\widetilde{T}=\bigcup_{P\in T} B(P,\ m\,|\,\chi(P)\,|^{-1})\subset D. \tag{4.22}$$

We then have

LEMMA 1. *The set $\widetilde{T}$ is single-sheeted.*

PROOF. Suppose that the set $\widetilde{T}$ is not single-sheeted, i.e., that there exist two distinct points $P_1,\ P_2 \in \widetilde{T}$ such that $\eta P_1 = \eta P_2$. Then, by the definition (4.22) of the set $\widetilde{T}$ as a union of balls, there exist points $P_{10},\ P_{20} \in T$ such that

$$P_1 \in B\left(P_{10},\ m\,|\,\chi(P_{10})\,|^{-1}\right)=B_1;\ P_2 \in B\left(P_{20},\ m\,|\,\chi(P_{20})\,|^{-1}\right)=B_2.$$

By assumption, ηS is a convex domain; therefore the points P_{10} and P_{20} can be joined by a straight segment $s \in S \cup T$.

Consider the intersection $\eta B_1 \cap \eta B_2$; it is not empty since $\eta P_1 = \eta P_2$. We take subsets $A_1 \subset B_1,\ A_2 \subset B_2$ satisfying the condition

$$\eta A_1 = \eta A_2 = \eta B_1 \cap \eta B_2.$$

The sets A_1 and A_2 are single-sheeted and have the same projection; they must be identical if they have at least one common point. But the segment ηs has common points with the intersection $\eta B_1 \cap \eta B_2$. Consequently the segment s intersects the sets A_1 and A_2; all points that belong to $s \cap A_1$ and $s \cap A_2$ are common to both sets. Hence it follows that $A_1 = A_2$.

Further, it is obvious that $P_1 \in A_1,\ P_2 \in A_2$ (since $\eta P_1 = \eta P_2$). Since $A_1 = A_2$ and the projections of these points coincide, we must conclude that $P_1 = P_2$.

Our supposition turns out to be invalid. The lemma is proved.

From the lemma just proved it follows that: since the domain $\widetilde{T}$ is single-sheeted, it coincides with its projection

$$\eta\widetilde{T}=\left\{\sum_{k=1}^{n} |\,z_k - z_k^0\,|^2 < m^2\,|\,\chi(z^0)\,|^{-2},\ z^0 \in \eta T\right\}.$$

Every function $f(P)$ holomorphic in the domain D is holomorphic for $P \in \widetilde{T}\subset D$; there corresponds to it in the domain $\eta\widetilde{T}$ the holomorphic function $f(P) \circ \eta^{-1} = f(z)$. In view of Theorem 17.3 this function is continuable to all points of the set

$$G = \left\{ \sum_{k=1}^{n} |z_k - z_k^0|^2 < m^2 |\chi(z^0)|^{-2},\ z^0 \in \eta S \right\}.$$

Consider the collection of the points of the domain D situated over the set G. This collection, in general, consists of some connected components. We denote by $\tilde{G}$ the component that contains the domain S. All functions $f(z)$, where $f \in \mathfrak{H}_D$, are analytically continued to points of the single-sheeted set $G > \tilde{G}$; consequently the functions $f \in \mathfrak{H}_D$ do not separate the points of the set $\tilde{G}$ with equal coordinates. We identify such points of the set $\tilde{G}$; we denote by $\hat{G}$ the set obtained from $\tilde{G}$ after this identification. Two cases are possible:

$$1)\ \eta\hat{G} \neq G; \quad 2)\ \eta\hat{G} = G.$$

In the first case we may replace $\tilde{G} \subset D$ by any domain $\hat{\hat{G}} \supset \hat{G}$ satisfying the condition $\eta\hat{\hat{G}} = G$; in the second case we take $\hat{\hat{G}} = \hat{G}$.

As a result of this replacement the domain D turns into a point set D'. We shall call the passage from the domain D to the set D' *Bremermann's c-process*. Evidently every function $f(P)$ that is holomorphic in the domain D is analytically continuable to all points of the set D'.

Let points $P_1, P_2 \in D$, while $P_1 \neq P_2$ and $\eta P_1 = \eta P_2$; let $U_1, U_2 \subset D'$ be (single-sheeted) neighborhoods of these points. If these neighborhoods have some open set in common, we identify all the points of the neighborhoods U_1 and U_2 having equal ground points. As a result the set D' turns into a certain set D''. We shall call the passage from the set D' to the set D'' *Bremermann's i-process*.

We again apply Bremermann's *i*-process, if possible, to the set D'' and so forth. After such an iteration we ultimately obtain a domain which does not permit the further application of the *i*-process. It is easy to see that this domain does not depend on the choice of pairs of points whose neighborhoods were identified in the corresponding *i*-process. We denote the domain thus obtained by $D \overline{\bigcup} \hat{\hat{G}}$.

Evidently every function holomorphic in the domain D is analytically continuable to all points of the set $D \overline{\bigcup} \hat{\hat{G}}$.

We shall call the passage from the domain D to the domain $D \overline{\bigcup} \hat{\hat{G}}$ *Bremermann's ci-process* or *Bremermann's process of holomorphic extension of the domain D*.

If the domain D is single-sheeted and the intersection $D \cap G$ is connected,

then $D \overline{\bigcup} \hat{\hat{G}} = D \bigcup G$; if the domain D is single-sheeted, while the intersection $D \bigcap G$ consists of some connected components, its holomorphic extension $D \overline{\bigcup} \hat{\hat{G}}$, generally speaking, turns out to be multiple-sheeted, as already remarked above.

5. **Iteration of Bremermann's process of holomorphic extension.** Let some domain D be given. There arises the problem: by applying Bremermann's process of holomorphic extension first to the domain D, and then to the domains obtained from it by this process, is it possible to construct a sequence of domains $\{D_\nu\}$, $\nu = 0, 1, 2, \cdots$, $D_0 = D$, $D_\nu \subset D_{\nu+1}$, converging to the holomorphy hull $H(D)$?

We consider this problem under the assumption that *the domain D, the domains appearing in all intermediate stages of its holomorphic extension, and the holomorphy hull $H(D)$ are all single-sheeted.*

THEOREM 17.5 (Bremermann [1], (I)). *If a domain $D \subset C^n$ remains unchanged by the application of Bremermann's process of holomorphic extension, however the auxiliary domain S may be chosen, then this domain D is a domain of holomorphy.*

PROOF. In the case at hand for $z \in S$

$$d_D(z) \geqslant m\,|\chi(z)|^{-1}.$$

Therefore

$$\inf_{z \in \partial S} d_D(z)\,|\chi(z)| = \inf_{z \in \bar{S}} d_D(z)\,|\chi(z)|. \tag{4.23}$$

We shall show that under our conditions $-\ln d_D(z)$ is a plurisubharmonic function.

It is well known (see the supplement to Theorem 13.4) that the function $-\ln d_D(z)$ is continuous; therefore one only needs to establish that the function $V(\lambda) = -\ln d_D(z^0 + \lambda b)$ is subharmonic in the open set $D \bigcap \{z_k = z_k^0 + \lambda b_k,\ k = 1, \cdots, n\}$ for any point $z^0(z_1^0, \cdots, z_n^0) \in D$ and any unit vector $b(b_1, \cdots, b_n)$.

We shall argue by contradiction. Suppose that the function $V(\lambda)$ is not subharmonic in some disk of radius r lying on the given plane in the domain D. We may always assume that the center of this disk coincides with the point z^0.

Then there exists a function $h(\lambda)$ harmonic for $|\lambda| < r$ and continuous for $|\lambda| \leq r$, such that

$$V(\lambda) \leqslant h(\lambda) \quad \text{for} \quad |\lambda| = r, \tag{4.24}$$

$$V(\lambda^0) > h(\lambda^0), \tag{4.25}$$

where λ^0 is some number satisfying the condition $|\lambda^0| < r$.

Let $\gamma(\lambda)$ be a harmonic polynomial approximating the harmonic function $h(\lambda)$; we take this polynomial so close to the function $h(\lambda)$ that inequalities (4.23) and (4.25) are valid for it. Then we form the harmonic polynomial $\gamma^*(\lambda)$, conjugate to the polynomial $\gamma(\lambda)$, and the polynomial $g(\lambda) = \gamma(\lambda) + i\gamma^*(\lambda)$ of the complex variable λ.

Next, by means of the equality

$$g(z) = g(z^0 + \lambda b + \lambda_2 b^2 + \cdots + \lambda_n b^n) = g(\lambda)$$

we define a function $g(z)$ holomorphic in the whole space C_z^n and therefore in the domain D. Here $b, b^2, \cdots, b^n$ are a system of orthonormal vectors constituting a basis in the space C^n, and $\lambda_2, \cdots, \lambda_n$ are complex parameters. Evidently, on the plane $z_k = z_k^0 + \lambda b_k$ $(k = 1, \cdots, n)$ the function $g(z)$ coincides with the function $g(\lambda)$ defined previously.

Take $S = \{z_k = z_k^0 + \lambda b_k;\ k = 1, \cdots, n;\ |\lambda| < r\}$ and $\chi(z) = \exp[g(z)]$. Then from inequality (4.24) for $\operatorname{Re}[g(z)]$ we find that at $z \in \partial S$

$$-\ln d_D(z) \leq \ln |\chi(z)|,$$

or

$$1 \leq d_D(z)\,|\chi(z)|. \tag{4.24$_1$}$$

On the other hand, from inequality (4.25) for $\operatorname{Re}[g(z)]$ we find that

$$1 > d_D(z^1)\,|\chi(z^1)|, \tag{4.25$_1$}$$

where the point $z^1 = z^0 + \lambda^0 b \in S$.

Inequalities (4.24_1) and (4.25_1) taken together are inconsistent with equality (4.23). We have been led to a contradiction and must reject our supposition.

Thus we have proved that $-\ln d_D(z)$ is a plurisubharmonic function. Hence in view of Theorems 13.5 and 14.3 (for the case C^n, where n is any natural number larger than unity) the assertion of Theorem 17.5 follows.

Now we turn to the construction of the sequence of domains $\{D_\nu\}$ having the properties mentioned at the beginning of the present subsection. First we assume that D is a bounded domain. We consider a mesh Q_δ of points $z^{\mu,\nu} = (z^{\mu_1,\nu_1}, \cdots, z^{\mu_n,\nu_n})$, where

$$z^{\mu_j,\nu_j} = \delta(\mu_j + i\nu_j).$$

Here μ_j, ν_j are arbitrary natural numbers and $\delta > 0$ is a certain number characterizing the closeness of the mesh. Evidently, in the domain D there lie a finite set of points $z^{\mu,\nu}$. We further take a system of K^n vectors

$$a^\rho = \left(\frac{1}{\sqrt{n}} e^{\frac{2\pi}{K} i\rho_1}, \ldots, \frac{1}{\sqrt{n}} e^{\frac{2\pi}{K} i\rho_n}\right), \tag{4.26}$$

where $\rho_j = 1, \cdots, K$; $j = 1, \cdots, n$, and K is some natural number. Through each point $z^{\mu,\nu} \in D$ we draw K^n one-complex-dimensional analytic planes containing the vectors (4.26). Let $R_{\mu,\nu;\rho}$ be the radius of the largest disk with its center at the point $z^{\mu,\nu} \in D$ lying in the analytic plane defined by the vector a^ρ. In each such plane we construct other $N-1$ disks, with centers at the points $z^{\mu,\nu}$, of radius

$$r_\sigma = \frac{\sigma}{N} R_{\mu,\nu;\rho}, \quad \text{for } \sigma = 1, \ldots, N-1;$$

N is a certain natural number.

From these disks (with centers at all the points $z^{\mu,\nu} \in D$) we make up a finite sequence $S_1, \cdots, S_{\alpha_1}$ in any way.

We apply the process of holomorphic extension to the domain D by taking $S = S_1$ and we denote the domain obtained by D_1. Since $S_2 \subset D \subseteq D_1$, we shall be able to apply the process of holomorphic extension by putting $S = S_2$ and so forth. In this way we extend the domain D to some domain D_{α_1}.

Next, in the domain D_{α_1} we construct a set of disks $S_{\alpha_1+1}, \cdots, S_{\alpha_2}$ by taking the mesh $Q_{\delta/2}$ instead of the mesh Q_δ, and replacing the numbers K and N by the numbers $2K$ and $2N$. After this we extend the domain D_{α_1} to some domain D_{α_2} and so forth.

Consider the sequence of domains $\{D_\nu\}$ we have constructed. Evidently $D_\nu \subseteq D_{\nu+1}$ and $D_\nu \subseteq H(D)$. Therefore the $\lim_{\nu\to\infty} D_\nu = D^* \subseteq H(D)$ necessarily exists. We shall show that D^* *is a domain of holomorphy and accordingly coincides with the holomorphy hull* $H(D)$ (since $H(D)$ is the smallest domain of holomorphy that contains the domain D).

In view of Theorem 17.5 our assertion will be proved if we show that the domain D^* cannot be enlarged by Bremermann's method of holomorphic extension. We shall argue by contradiction. Suppose that such an enlargement of the domain D^* is possible and corresponds to the choice of some disk S^* as an auxiliary domain.

Take a number α_0 so large that, for $\alpha > \alpha_0$, the domain D_α differs sufficiently little from the domain D^*. Choose a number $\alpha_{00} > \alpha_0$ such that the disk $S_{\alpha_{00}}$ differs sufficiently little (in the sense specified below) from the disk S_0. This latter condition is possible since the set of the disks S_ν is dense in the set of all disks lying within the limits of the domain D^* in different analytic planes. Then Bremermann's process of holomorphic extension, when it is applied to the domain $D_{\alpha_{00}-1}$ for the auxiliary disk $S_{\alpha_{00}}$, gives a domain $D_{\alpha_{00}}$ sufficiently close to the domain obtained by applying this process to the domain D^* for the auxiliary domain S^*. The numbers α_0 and α_{00} are to be so chosen that in view of what has been said the domain $D_{\alpha_{00}}$ falls beyond the domain D^*. This is impossible and therefore we must reject the supposition we have made. The proof is complete.

Now assume that $D \subset C^n$ is a unbounded domain. Then we consider some sequence of domains $D^{(k)} \longrightarrow D$ as $k \longrightarrow \infty$. Next we construct a sequence of domains $D_\nu^{(k)} \longrightarrow H(D^{(k)})$ as $\nu \longrightarrow \infty$. Finally we form the sequence of domains $\{D_\nu^{(\nu)}\}$. By means of Theorem 6.3 (Behnke-Stein) for unbounded domains it can be proved that $\lim_{\nu\to\infty} D_\nu^{(\nu)} = H(D)$.

Another process of holomorphic extension of domains $D \subset C^n$ (for which also $H(D) \subset C^n$) also converging to the domain $H(D)$ is proposed by Taylor [1].

In conclusion we remark that, since the general method of holomorphic extension is inconvenient in practice, the methods of holomorphic extension applicable to individual classes of domains are of great value.

§18. HOLOMORPHIC EXTENSION OF SEMITUBULAR DOMAINS

1. Semitubular domains.

DEFINITION. A domain D of the space $C^2_{w,z}$ of the variables $w = u + iv$, $z = x + iy$ is said to be *semitubular* if

$$D = \mathfrak{D} \times R_1^{(v)}.$$

Here $\mathfrak{D}$ is some domain in the space $R_3^{(u,x,y)}$, while $R_1^{(v)}$ is, as usual, the space (axis) of the variable v. The domain $\mathfrak{D}$ is called the *base* of the semitubular domain D.

It follows from the above definition that a semitubular domain has the automorphism

$$z = z^0, \quad w = w^0 + it.$$

This property may be taken as the basis of the definition of a semitubular domain.

We may consider a semitubular domain over the space C^2 of the variables w, z. To obtain such a domain, one needs to replace the domain $\mathfrak{D} \subset R_3^{(u, x, y)}$ by a domain $\mathfrak{D} \subset Z \times R_1^{(u)}$, where Z is a Riemann domain over the plane C_z^1 of the variable z having no interior branch points (see Bremermann [2]).

We agree to say that the semitubular domain D has a base of *normal type* if its base $\mathfrak{D} \subset R_3^{(u, x, y)}$ can be given by a condition of the form

$$\mathfrak{D} = \{z \in G, \; V_1(z) < u < V_2(z)\},$$

where $V_1(z)$, $V_2(z)$ are certain functions given in a domain $G \subset C_z^1$. In other words, in this case the intersection of the domain $\mathfrak{D}$ with straight lines parallel to the axis Ou is always either empty or reduced to one segment.

2. Holomorphy hull of a semitubular domain with a base of normal type.

THEOREM 18.1 **(Bremermann [2], Stein [1]).** *In order that the semitubular domain D with a base of normal type should be a domain of holomorphy, it is necessary and sufficient that the function $V_1(z)$ should be subharmonic, while the function $V_2(z)$ should be superharmonic in the domain G.*

PROOF. I. Necessity. 1) Evidently $V_1(z) < \infty$, while $V_2(z) > -\infty$ since the values of the variable u, being coordinates of points of the domain $\mathfrak{D} \subset R_3^{(u, x, y)}$, cannot be infinite.

2) Since $\mathfrak{D}$ is a domain, along with each point $(z_0, u_0) \in \mathfrak{D}$ its (three-dimensional) neighborhood belongs to this domain. Consequently, the inequality $V_1(z) < u_0 < V_2(z)$ holds in a certain (two-dimensional) neighborhood of the point z_0. This means that at any point $z_0 \in G$ the function $V_1(z)$ is upper semicontinuous, while the function $V_2(z)$ is lower semicontinuous.

3) Now suppose that the function $V_1(z)$ will not be subharmonic in any neighborhood of that point in G.

Since $V_1(z) < V_2(z)$ at every point $z \in G$, there exists a number $\epsilon > 0$ such that $V_1(z_0) + 2\epsilon = V_2(z_0)$. Because of the semicontinuity of the functions $V_1(z)$ and $V_2(z)$ in the domain G there exists a neighborhood U_{z_0} of the point z_0 such that $V_1(z) < V_1(z_0) + \epsilon$; $V_2(z) > V_2(z_0) - \epsilon = V_1(z_0) + \epsilon$ for $z \in U_{z_0}$. Hence it follows that

$$\{z \in U_{z_0}, \; V_1(z) < u < V_1(z_0) + \epsilon\} \subset \mathfrak{D}.$$

Since the function $V_1(z)$ is not subharmonic in the domain U_{z_0}, there exists a disk $K \Subset U_{z_0}$ and a function $h(z)$, harmonic in this disk and continuous in the closed disk $\bar{K}$, such that $V_1(z) \leq h(z)$ for $z \in \partial K$, but such that this inequality is violated in the disk K itself. Let the quantity $V_1(z) - h(z)$ take on its maximum in the disk K at a point $z_1 \in K$ and $V_1(z_1) - h(z_1) = d > 0$. Then

$$h(z) + d \geq V_1(z) \text{ for } z \in K,$$

$$h(z) + d > V_1(z) \text{ for } z \in \partial K.$$

Let $h^*(z)$ be a harmonic function in the disk K, conjugate to the function $h(z)$. We consider the analytic surface

$$(F) \qquad w = h(z) + d + ih^*(z), \quad z \in K.$$

Evidently it passes through the point (w_1, z_1), where $w_1 = h(z_1) + d + ih^*(z_1)$. The point $(w_1, z_1) \in \partial D$ since $h(z_1) + d = V(z_1)$.

Since the closed disk $\bar{K} \subset U_{z_0}$, where $V_1(z) < V_1(z_0) + \epsilon$, we can choose a number $\epsilon' > 0$ such that

$$V_1(z) < V_1(z_0) + \epsilon - \epsilon' \text{ for } z \in \bar{K}.$$

Since $h(z_1) + d = V_1(z_1)$, we have

$$h(z_1) + d < V_1(z_0) + \epsilon - \epsilon'.$$

Consider the domain

$$K' = \{h(z) + d < V_1(z_0) + \epsilon - \epsilon'\} \cap K.$$

The foregoing inequality implies that the point $z_1 \in K'$.

We shall show that a point $z \in \partial K'$ cannot belong to the disk K. Indeed, if such a point $z \in K$, then

$$h(z) + d = V_1(z_0) + \epsilon - \epsilon',$$

which is impossible, since for $z \in \bar{K}$

$$V_1(z) < V_1(z_0) + \epsilon - \epsilon'.$$

Therefore $\partial K' = \partial K$ and at a point $z \in \partial K'$

$$h(z) + d > V_1(z).$$

On the other hand, at the same point

$$h(z) + d \leq V_1(z_0) + \epsilon - \epsilon' < V_1(z_0) + \epsilon.$$

It follows from the last two inequalities that to every point $z \in \partial K'$ on the surface (F) there corresponds a point $(w, z) \in D$. We have observed above that the point $(w_1, z_1) \in (F)$ is a boundary point of the domain D. Finally, it is obvious that the surface (F) may be approximated by the analytic surfaces

$$(F_k) \qquad w = h(z) + d + \delta_k + ih^*(z), \qquad z \in K,$$

where δ_k decreases monotonically and tends towards zero as $k \longrightarrow \infty$. It is easy to see that $(\bar{F}_k) \subset D$ for $\delta_k < \epsilon'$.

By the Behnke-Sommer Theorem 11.9, (I), the enumerated properties of the surfaces (F) and (F_k) cannot hold if the domain D is a domain of holomorphy. Thus our supposition is eliminated and the subharmonicity of the function $V_1(z)$ is proved.

The proof for the superharmonicity of the function $V_2(z)$ is carried out in a completely analogous manner.

II. Sufficiency. Under the assumption of the theorem the functions $V_1(z) - u$ and $u - V_2(z)$ are shown to be plurisubharmonic in the domain D. This fact is easy to prove on the basis of Theorem 13.1.

On the basis of Theorem 13.6 and by the use of the fact that the intersection of domains convex in the sense of Hartogs again consists of domains convex in the sense of Hartogs, we establish that the domain D is convex in the sense of Hartogs. After this the final result will follow from Oka's Theorem 14.3.

COROLLARY. *The holomorphy hull of a semitubular domain D with a base of normal type*

$$\mathfrak{D} = \{z \in G,\ V_1(z) < u < V_2(z)\}$$

is a semitubular domain with a base of normal type

$$\tilde{\mathfrak{D}} = \{z \in G,\ W_1(z) < u < W_2(z)\}.$$

Here $W_1(z)$ is the greatest subharmonic minorant of the function $V_1(z)$, and $W_2(z)$ the least superharmonic majorant of the function $V_2(z)$.

This proposition easily follows from Theorem 18.1, if one considers that in view of Theorem 13.7, (I) the holomorphy hull of the semitubular domain is again a semitubular domain.

The domain $\tilde{\mathfrak{D}}$ is called the *sub-superharmonic* hull of the domain $\mathfrak{D}$.

REMARK 1. Evidently, to the sub-superharmonic hull $\tilde{\mathfrak{D}}$ of the domain $\mathfrak{D}$ there belong points obtained in the following way: Let $\Pi = \{z = z(t), u = u(t)\} \subset \mathfrak{D}$ be any closed line whose projection $\Pi' = \{z = z(t), u = 0\} \Subset G$ bounds a simply-connected domain $\pi \Subset G$. Here t is a parameter by means of which the lines Π and Π' are given. In the domain π we find a harmonic function $h(z)$ satisfying the boundary condition

$$h(z)\,|_{\Pi'} = u(t).$$

Then the surface $\{u = h(z)\}$ passes through the line Π. By adjoining to the domain $\mathfrak{D}$ the part of this surface lying inside the contour Π (together with a sufficiently small neighborhood of it), we extend the domain $\mathfrak{D}$.

The corresponding enlargement of the semitubular domain D will evidently be its holomorphic extension.

REMARK 2. Theorem 18.1 remains valid for multiple-sheeted semitubular domains. However in this case the proof of the sufficiency of the above condition requires new tools (see Bremermann [2]).

3. **Locally normal hull.** Let $\mathfrak{D}$ be a domain in a Riemann domain $Z \times R_1^{(u)}$, where Z is, as above, a Riemann domain over the plane C_z^1. It is assumed that the domains Z and $\mathfrak{D}$ have no interior branch points.

Then there corresponds to each point $(u_0, z_0) \in \mathfrak{D}$, where $z_0 \in Z$ and $u_0 \in R_1^{(u)}$, a single-sheeted neighborhood $\mathfrak{U}_0 \subset Z$ of the point z_0 with the following properties:

1) $\mathfrak{U}_0$ is a disk;

2) if a point $z \in \mathfrak{U}_0$, then the point $(u_0, z) \in \mathfrak{D}$;

3) $\mathfrak{U}_0$ is the maximum disk having the property 2).

We consider the set of points $(u, z) \in \mathfrak{D}$ for which the point $z \in \mathfrak{U}_0$. We denote by $U(\mathfrak{D}, u_0, z_0)$ the connected component of this set which contains the point (u_0, z_0).

We agree to call $\mathfrak{D}$ a *base of locally normal type* if for each point $(u, z) \in \mathfrak{D}$ the corresponding set $U(\mathfrak{D}, u, z)$ turns out to be a base of normal type.

EXAMPLE. Let $Z = C_z^1$, $\mathfrak{D} = \{1 < |z| < 2, \operatorname{Arg} z - 1/2 < u < \operatorname{Arg} z + 1/2\}$ (we have already considered such a domain in §13.5, Chapter II, (I)). Take $z_0 = 1.5$; $u_0 = 0$. Then $\mathfrak{U}_0 = \{|z - 1.5| < 1/2\}$, and the set $\mathfrak{D} \cap \{|z - 1.5| < 1/2\}$ consists of an infinite set of connected components; the one that contains the point with coordinates $z_0 = 1.5$, $u_0 = 0$ plays the role of the domain $U(\mathfrak{D}, u_0, z_0)$. Analogously

there are domains $U(\mathfrak{D}, u, z)$ for other points $(u, z) \in \mathfrak{D}$.

Thus the domain $\mathfrak{D}$ is shown to be a locally normal base. However it is not a normal base. The smallest normal base containing the domain $\mathfrak{D}$ is the domain $\{1 < |z| < 2\} \times R_1^{(u)}$.

THEOREM 18.2. *A domain $\mathfrak{D} \subset Z \times R_1^{(u)}$, being a locally normal base, may always be defined by the condition*

$$\mathfrak{D} = \{z \in G,\ V_1(z) < u < V_2(z)\}.$$

Here G is a Riemann domain over the plane C_z^1, $V_1(z)$ and $V_2(z)$ are single-valued functions defined in the domain G.

REMARK. The domain G does not, in general, coincide with any part of the domain A. Thus in the above example $Z = C_z^1$, but the domain G is the corresponding part of the Riemann surface of the function Ln z.

PROOF. By assumption, every domain $U(\mathfrak{D}, u_0, z_0)$ is a base of normal type. Any straight line $z = \text{const}$ intersecting this domain cuts from it one connected segment. The values of the coordinates u at the lower and upper points of that segment define the functions $u = V_1(z)$ and $u = V_2(z)$. As a result (still by taking account of the fact that the domain $\mathfrak{U}_0$ is single-sheeted and we can identify it with its projection $\underline{\mathfrak{U}}_0 \subset C_z^1$) we specify the domain $U(\mathfrak{D}, u_0, z_0)$ by the relation

$$U(\mathfrak{D}, u_0, z_0) = \{\underline{z} \in \underline{\mathfrak{U}}_0,\ V_1(\underline{z}) < u < V_2(\underline{z})\}.$$

If $U_k = U(\mathfrak{D}, u_k, z_k)$, $k = 1, 2$, are two such domains, and $V_1^{(k)}(\underline{z})$, $V_2^{(k)}(\underline{z})$, $k = 1, 2$, are corresponding functions in the above relation and $U_1 \cap U_2 \neq \emptyset$, then

$$V_j^{(1)}(\underline{z}) = V_j^{(2)}(\underline{z}), \quad j = 1, 2, \qquad \text{for} \quad z \in U_1 \cap U_2.$$

Because of this property of the functions $V_j(\underline{z})$, $j = 1, 2$, defined in the different domains $U(\mathfrak{D}, u, z)$, we may unite them into a single-valued function $V_j(z)$ given in some Riemann domain G over the plane C_z^1. This uniting process proceeds in exactly the same way as the process of analytically continuing a holomorphic functional element to obtain a domain of definition of an analytic function.

By means of the functions $V_j(z)$, $j = 1, 2$, the domain $\mathfrak{D}$ can be represented in the required form. The theorem is proved.

THEOREM 18.3. *For every domain $\mathfrak{D} \subset Z \times R_1^{(u)}$ there exists the (uniquely*

defined) *smallest locally normal domain* $\mathfrak{D}^* \subset Z \times R_1^{(u)}$ *containing the domain* $\mathfrak{D}$.

This domain $\mathfrak{D}^*$ is called *the locally normal hull of the domain* $\mathfrak{D}$.

PROOF. We use the obvious fact that for every domain $U(\mathfrak{D}, u, z)$ we can construct the smallest normal domain $\widetilde{U}(\mathfrak{D}, u, z)$ containing it. We extend the domain $\mathfrak{D}$ by replacing each domain $U(\mathfrak{D}, u, z)$ by the corresponding domain $\widetilde{U}(\mathfrak{D}, u, z)$. In this connection we shall regard points of the domain $U(\mathfrak{D}, u_k, z_k)$, $k = 1, 2$, with identical coordinates as distinct or coinciding according as the intersection $U(\mathfrak{D}, u_1, z_1) \cap U(\mathfrak{D}, u_2, z_2)$ is empty or not. [1)]

Denote by $\mathfrak{D}_1$ the domain obtained by extending the domain $\mathfrak{D}$. If $\mathfrak{D}_1$ is not a locally normal domain, we again apply to it the above-mentioned process of extension. In that case, of course, it is sufficient to consider the domain $U(\mathfrak{D}_1, u, z)$ only for points $(u, z) \in \mathfrak{D}_1 \setminus \mathfrak{D}$. The domain obtained as a result of this extension is denoted by $\mathfrak{D}_2$, and so forth.

Then two cases are possible: for some number $k > 0$ it turns out that $\mathfrak{D}_k = \mathfrak{D}_{k+1}$ (where $\mathfrak{D}_0 = \mathfrak{D}$), or else we obtain an infinite sequence of domains $\{\mathfrak{D}_\nu\}$. In the first case we put $\mathfrak{D}^* = \mathfrak{D}_k$, in the second $\mathfrak{D}^* = \lim_{\nu\to\infty} \mathfrak{D}_\nu$ (this limit exists since $\mathfrak{D}_\nu \subset \mathfrak{D}_{\nu+1}$).

In our process of construction the domain $\mathfrak{D}^* \supset \mathfrak{D}$ is defined uniquely; it is a locally normal domain. Therefore our assertion will be proved if we establish that every other locally normal domain ${}^*\mathfrak{D} \supset \mathfrak{D}$ contains the domain $\mathfrak{D}^*$.

This is obvious when $\mathfrak{D}^* = \mathfrak{D}$. If $\mathfrak{D}^* \neq \mathfrak{D}$, we take into account the fact that points added to the domain $\mathfrak{D}$ on the passage to the domain $\mathfrak{D}_1$ must belong to any locally normal domain containing the domain $\mathfrak{D}$. Therefore $\mathfrak{D}_1 \subseteq {}^*\mathfrak{D}$ and, in general, $\mathfrak{D}_\nu \subseteq {}^*\mathfrak{D}$ for any $\nu \geq 1$. Hence we conclude that $\lim_{\nu\to\infty} \mathfrak{D}_\nu = \mathfrak{D}^* \subseteq {}^*\mathfrak{D}$ as well. Theorem 18.3 is proved.

4. Holomorphic extension of a semitubular domain of general type.

THEOREM 18.4 (Bremermann [2]). *Let* $D = \mathfrak{D} \times R_1^{(v)}$ *be a semitubular domain with a base* $\mathfrak{D} \subset Z \times R_1^{(u)}$ *having no interior branch points. Construct:*

1) *the locally normal hull* $\mathfrak{D}^*$ *of the domain* $\mathfrak{D}$;

2) *the sub-superharmonic hull* $\widetilde{\mathfrak{D}}$ *of the domain* $\mathfrak{D}^*$.

Then the holomorphy hull $H(D) = D$, *where* D *is the semitubular domain with the base* $\widetilde{\mathfrak{D}}$.

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1) This situation shows visually how a multiple-sheeted holomorphy hull of a single-sheeted semitubular domain is obtained.

REMARK. As regards the definition of subharmonic and superharmonic functions on a Riemann surface, see §13.8. The plurisubharmonic function defined in the case when the dimension of a complex space R is equal to unity is reduced to a subharmonic function on a Riemann surface.

Theorem 18.4 will follow from Theorems 18.1 (extended to the domain over the space C^2) and 18.2 if we show that every function holomorphic in the domain D is analytically continuable to the whole semitubular domain D^* with base $\mathfrak{D}^*$. We shall omit the proof of this statement.

We note that the above methods of constructing the domain $\mathfrak{D}^*$ (Theorem 18.3) and $\widetilde{\mathfrak{D}}$ (Remark 1 to Theorem 18.1) in general require an infinite number of steps. After a finite number of similar steps we obtain as a rule only some holomorphic extension of the semitubular domain D.

EXAMPLES. I. Let $\mathfrak{D}$ be a torus in the space $R_3^{(x,\,y,\,u)}$, defined by the condition

$$\mathfrak{D} = \{(|z| - R)^2 + u^2 < r^2\},$$

where $r < R/2$. The symmetry axis of this torus is the axis Ou and $R - r < |z| < R + r$ at points on the torus. In this case $V_{1,2}(z) = \mp\sqrt{r^2 - (|z| - R)^2}$. We further find that so long as $r < R/2$

$$\Delta V_2 = \frac{1}{V_2} - \frac{(R-|z|)^2}{V_2^2} - \frac{R-|z|}{|z|\,V_2} < 0,$$

while $\Delta V_1 < 0$ for $R - r < |z| < R + r$. Therefore $\widetilde{\mathfrak{D}} = \mathfrak{D}^* = \mathfrak{D}$.

II. Let $\mathfrak{D}$ be a torus in the space $R_3^{(x,\,y,\,u)}$, whose symmetry axis coincides with the axis Ox.

In this case the locally normal hull $\mathfrak{D}^*$ coincides with the convex hull $O(\mathfrak{D})$ of the torus $\mathfrak{D}$, i.e.,

$$\mathfrak{D}^* = \{|x| < r + R,\ |y| < r,\ V_1(z) < u < V_2(z)\}.$$

Here R and r have the same meaning as in the preceding example; $V_1(z)$ and $V_2(z)$ are convex functions for the values of z under consideration.

It is well known that a convex function is always a subharmonic function (this is a special case of the similar proposition in the theory of plurisubharmonic functions; see §13.1). Therefore the sub-superharmonic hull of the domain $\mathfrak{D}^*$ coincides with $\mathfrak{D}^*$ itself. Thus $\widetilde{\mathfrak{D}} = \widetilde{\mathfrak{D}}^* = \mathfrak{D}^*$.

In Bremermann's paper [2] one can also find more complicated examples when the locally normal hull of a single-sheeted domain $\mathfrak{D}$ turns out to be multiple-sheeted. In such a case the holomorphy hull of the corresponding single-sheeted semitubular domain D is also naturally a multiple-sheeted domain.

In conclusion we remark that the results obtained may be extended to the case of the semitubular sets of the space $C^2_{w,z}$ whose base is a line or a surface in the space $R_3^{(u,x,y)}$.

For example, if $\mathfrak{D} = \{u^2 + x^2 = r^2,\ y = 0\}$, then $\widetilde{\mathfrak{D}} = \{u^2 + x^2 \le r^2,\ y = 0\}$. In this case the dimension of $\widetilde{\mathfrak{D}}$ is larger by unity than that of $\mathfrak{D}$; the analogous fact, of course, is true for the sets D and $H(D)$. Moreover it is easy to construct examples of lines $\mathfrak{D}$ in the space $R_3^{(u,x,y)}$ for which $\widetilde{\mathfrak{D}} = \mathfrak{D}$.

It is of interest to note that if $\mathfrak{D}$ is a two-dimensional surface in the space $R_3^{(u,x,y)}$, then on the passage to the set $\widetilde{\mathfrak{D}}$ the dimension always increases by unity, excepting the case when $\mathfrak{D} = \{u = h(z),\ z \in G\}$, where $h(z)$ is a harmonic function in the domain G. In this last case $\mathfrak{D} = \widetilde{\mathfrak{D}}$ (see Bremermann [2], Stein [1]).

§19. HOLOMORPHIC EXTENSION OF DOMAINS OF SPECIAL TYPE

1. The "edge of the wedge" theorem. Formulation and beginning of the proof. The "edge of the wedge" theorem represents an interesting extension of the equivalence principle of continuous and analytic continuations (see §6.7, Chapter I, (I)). It was first obtained by N. N. Bogoljubov in connection with the foundation of the dispersion relations in quantum field theory. However, this theorem itself and also its various generalizations are of general interest, since they establish the possibility of the extension of domains of a definite type. We note that there have been many publications dealing with the "edge of the wedge" theorem both in the physical journals and in monographs.

Let us introduce some notation and terminology: Let T^+, T^- be radial tubular domains in the space C^n_z of the complex variables $z_k = x_k + iy_k$, $k = 1, \cdots, n$; their bases are (not necessarily convex) cones V^+, $V^- \subset R_n^{(\mathrm{Im})}$ with vertices at the origin of coordinates. Here $R_n^{(\mathrm{Im})}$ is the space of the real variables $y_1, \cdots, y_n$. As for the cones V^+ and V^-, it is assumed that $V^+ \cap (-V^-) \neq \emptyset$, where $-V^-$ is the cone defined by the following condition: a point $y(y_1, \cdots, y_n) \in -V^-$ if the the point $-y(-y_1, \cdots, -y_n) \in V^-$.

We shall call a cone $V' \subset R_n$ a subcone of a cone $V \subset R_n$ (both with vertices at the origin of coordinates) if $\bar{V}' \subset V \cup \{0\}$. We set

$$\Gamma^{\pm} = \{\pm y_1 > \sqrt{y_2^2 + \cdots + y_n^2}\}.$$

The cones Γ^+ and Γ^- are usually called the light cones.

By $O(G)$ we denote as above the open convex hull of an open set G.

In the following formulas $|y| = (y_1^2 + \cdots + y_n^2)^{1/2}$, $|z| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$, and $z\xi = z_1\xi_1 + \cdots + z_n\xi_n$ if $z \in C_z^n$, $\xi \in R_n^{(\xi)}$.

THEOREM 19.1 (the "edge of the wedge" theorem).[1] *If:*

1) *Functions $f^+(z)$ and $f^-(z)$ are holomorphic in tubular domains T^+ and T^-, respectively.*

2) *For any number $\epsilon > 0$ and any subcones $(V^+)'$ and $(V^-)'$ of cones V^+ and V^-*

$$|f^{\pm}(z)| \leqslant Ae^{\varepsilon|y|}|y|^{-\rho_\pm}(1+|z|)^{m_\pm} \tag{4.27}$$

for $z \in (T^{\pm})$. Here $(T^+)'$, $(T^-)' \subset C^n$ are tubular domains with bases $(V^+)'$, $(V^-)'$, the quantities $\rho_\pm$ and $m_\pm$ are independent of the choice of the number ϵ and the cones $(V^{\pm})$, and $A = A(\epsilon, (V^{\pm})')$.

3) *For any function $\phi(x) \in \mathcal{C}^\infty$ with a finite support contained in an open set G of the space $R_n^{(\mathrm{Re})}$ of the real variables $x_1, \cdots, x_n$, there exists a finite and uniform limit*

$$\lim_{y\to 0,\, y\in V^+} \int_{R_n^{(\mathrm{Re})}} f^+(z)\varphi(x)\,dx = \lim_{y\to 0,\, y\in V^-} \int_{R_n^{(\mathrm{Re})}} f^-(z)\varphi(x)\,dx. \tag{4.28}$$

Then there exists in the domain $T = H(T^+) \cup H(T^-) \cup \tilde{G}$ a holomorphic function $f(z)$ coinciding with the function $f^{\pm}(z)$, or in the domains T^+ and T^-, respectively. Here $\tilde{G}$ is the neighborhood of the set G in the space C^n defined by the equation:

$$\tilde{G} = \bigcup_{x'\in G} \{|z - x'| < \theta d_G(x')\},$$

where $d_G(x')$ is the distance of the point x' to the boundary of the domain G and θ a positive quantity depending only the choice of the cones $V^{\pm}$.

REMARK. Thus Theorem 19.1 establishes that the open set $T^+ \cup T^-$ may be

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1) See Vladimirov [3, 4]. These papers give further literature concerning Theorem 19.1.

extended to the domain T with respect to the collection of functions holomorphic on this set and satisfying the conditions (4.27) and (4.28).

PROOF. The possibility of the analytic continuation of the functions $f^{\pm}(z)$ to the tubular domains $H(T^{\pm})$ (having, in view of Theorem 13.14, the hulls $\mathfrak{H}(V^{\pm})$ as their bases) is obvious. It suffices to prove that these functions have identical analytic continuations to the domains $\widetilde{G}$ or, in other words, into a neighborhood of any point $x^0 \in G$. We take the point x^0 as the origin of coordinates in the space $R_n^{(\mathrm{Re})}$. Thus in the following argument we may put $G = \{|x| < \eta\}$, where the number $\eta > 0$ may be taken as small as desired.

Since $V^+ \cap V^- \neq \emptyset$, there exists a cone $\Gamma \subset [V^+ \cap V^-]$ which can be transformed into the cone Γ^+ by means of a rotation and a dilatation in the space $R_n^{(\mathrm{Im})}$. Therefore it is sufficient to prove Theorem 19.1 for the case of the light cones $\Gamma^{\pm}$; *we assume in what follows that these cones* $\Gamma^{\pm}$ *are used for the bases of the tubular domains* $T^{\pm}$.

From the theory of generalized functions there follows the following general fact. [1] If the function $f^+(z)$ is holomorphic for $z \in T^+$ and satisfies the condition (4.27), then for $y \to 0$ there exist a boundary value $f^+(x) \in S^*$. Here S^* is the space of generalized functions which is conjugate to the space S of basic functions $\phi(\xi)$ defined in the space $R_n^{(\xi)}$ of real variables $\xi_1, \cdots, \xi_n$ (see subsection 6 of the Introduction). This boundary value is the Fourier transform of a generalized function $g^+(\xi)$ vanishing outside the closed cone $\bar{\Gamma}^+ \subset R_n^{(\xi)}$:

$$f^+(x) = \int_{R_n^{(\xi)}} g^+(\xi)\, e^{i\xi x}\, d\xi. \tag{4.29}$$

Therefore Theorem 19.1 is true if the following theorem holds.

THEOREM 19.2. [2] *There are given generalized functions* $f^{\pm}(\xi)$ *which vanish outside the closed cones* $\bar{\Gamma}^{\pm}$ *(respectively). It is known that their Fourier transforms* $\tilde{f}^+(x)$ *and* $\tilde{f}^-(x)$ *are equal to each other in the ball* $\{|x| < \eta\}$.

Then there exists a function $f(z)$, *holomorphic in the domain* $E_\eta = \{|z_1|^2 + \sqrt{|z_2|^2 + \cdots + |z_n|^2} < \eta/2\sqrt{2}\}$ *and coinciding with given values* $f^{\pm}(x)$

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1) See L. Schwartz, *Transformation de Laplace des distributions*, Medd. Lunds Univ. (Suppl.), 1952, pp. 196–206, and Vladimirov [1, 4].

2) See N. N. Bogoljubov, B. V. Medvedev and M. K. Polivanov, *Problems of the theory of dispersion relations*, Fizmatgiz, Moscow, 1958, p. 152 ff. (Russian)

in the intersection of this domain with the plane $y_1 = \cdots = y_n = 0$ (*i.e., for real values* $z_k = x_k$, $k = 1, \cdots, n$). *In every domain* $D \Subset E_\eta$ *this function* $f(z)$ *allows the representation*

$$f(z) = \left(\tilde{f}^+(x'), H^+(z, x')\right) + \left(\tilde{f}^-(x'), H^-(z, x')\right). \tag{4.30}$$

Here $H^{\pm}(z, x')$ *are functions, chosen for the domain* D, *which are holomorphic in the domain* D *and take on values in the space* $S_{x'}$.

2. **Continuation of the proof of Bogoljubov's theorem.** Later on we shall need the

LEMMA 1. *Let a generalized function* $\tilde{f}(x) = \int_{R_n^{(\xi)}} f(\xi)\, e^{ix\xi}\, d\xi$ *vanish in the ball* $\{|x| < \eta\} \subset R_n^{(x)}$. *Consider the collection of the entire functions depending on a parameter* $\epsilon > 0$

$$\tilde{f}_\varepsilon(z) = \int_{R_n^{(\xi)}} f(\xi) \exp(iz\xi - \varepsilon|\xi|^2)\, d\xi = (\bar{f}(\xi), \exp(iz\xi - \varepsilon|\xi|^2)). \tag{4.31}$$

Then the function $f_\epsilon(z)$ *and all its derivatives tend uniformly towards zero in the ball* $\{|z| < \eta/\sqrt{2}\} \subset C_z^n$ *as* $\epsilon \to +0$.

PROOF. Let K be a compactum contained in the ball $\{|z| < \eta/\sqrt{2}\}$. Then there exists a number $\eta_1 < \eta$ such that $K \subset \{|z| \le \eta_1/\sqrt{2}\}$. Since $f(\xi) \in S^*$, while $\exp(-y\xi - \epsilon|\xi|^2) \in S$, we may regard the expression (4.31) as the Fourier transform of the product $f(\xi)\exp(-y\xi - \epsilon|\xi|^2)$. By a theorem on convolution we obtain

$$\begin{aligned} f_\varepsilon(z) &= \frac{1}{(2\pi)^n} \tilde{f} * \widetilde{\exp[-y\xi - \varepsilon|\xi|^2]} \\ &= \frac{1}{(4\pi\varepsilon)^{\frac{n}{2}}} \int_{R_n^{(x')}} \tilde{f}(x') \exp\left[-\frac{1}{4\varepsilon}\sum_{j=1}^{n}(z_j - x'_j)^2\right] dx'. \end{aligned}$$

Since $\tilde{f}(x') = 0$ for $|x'| < \eta$, then it is found that

$$f_\varepsilon(z) = \frac{1}{(4\pi\varepsilon)^{\frac{n}{2}}} \int_{R_n^{(x')}} \tilde{f}(x')\, \varphi(|x'|^2) \exp\left[-\frac{1}{4\varepsilon}\sum_{j=1}^{n}(z_j - x'_j)^2\right] dx', \tag{4.32}$$

where $\phi(t)$ is a fixed function of the class C^∞ of a real parameter t, which is equal to unity for $t \ge \eta_3$ and zero for $t \le \eta_2$. Here $0 < \eta_1 < \eta_2 < \eta_3 < \eta$.

For every point $z \in C_z^n$ the function

$$\psi_\varepsilon(z;\ x') = (4\pi\varepsilon)^{-\frac{n}{2}} \varphi(|x'|^2) \exp\left[-\frac{1}{4\varepsilon}\sum_{j=1}^{n}(z_j - x_j')^2\right]$$

belongs to the space $S_{x'}$. Therefore the convolution (4.32) may be regarded as the value of the functional $\tilde{f}$ on the basic function $\psi_\epsilon(z;\ x')$:

$$f_\varepsilon(z) = (\bar{\tilde{f}}(x'),\ \psi_\varepsilon(z;\ x')) = \overline{(\tilde{f}(x'),\ \overline{\varphi_\varepsilon(z;\ x')})}. \tag{4.33}$$

We shall show that $\overline{\psi_\epsilon(z;\ x')} \longrightarrow 0$ as $\epsilon \longrightarrow +0$ in the sense of the topology of the space S (see subsection 6 of the Introduction), while all norms $\|\overline{\psi_\epsilon(z;\ x')}\|_p$ tend uniformly to zero in the ball $\{|z| < \eta/\sqrt{2}\}$.

For let $|z| \leq \eta_1/\sqrt{2}$. Then, setting $\Delta = (\eta_2^2 - \eta_1^2)/8$, we have

$$\begin{aligned}
&\|\overline{\psi_\varepsilon(z;\ x')}\|_p \\
&\leqslant c_p'\varepsilon^{-\frac{n}{2}-p} \sup_{|x'|\geqslant\eta_2}\left\{|x'|^{2p}\exp\left[-\frac{1}{4\varepsilon}\operatorname{Re}\sum_{j=1}^{n}(\bar{z}_j - x_j')^2\right]\right\} \\
&= c_p'\varepsilon^{-\frac{n}{2}-p} \sup_{|x'|\geqslant\eta_2}\left\{|x'|^{2p}\exp\left[-\frac{1}{4\varepsilon}\left(\frac{1}{2}|x'|^2\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. -|z|^2 - \sum_{j=1}^{n}\left(\sqrt{2}x_j - \frac{1}{\sqrt{2}}x_j'\right)^2\right)\right]\right\} \\
&\qquad \leqslant c_p'\varepsilon^{-\frac{n}{2}-p} e^{-\frac{\Delta}{\varepsilon}} \sup_{|x'|\geqslant\eta_2}\left\{|x'|^{2p}\exp\left[-\frac{1}{8\varepsilon}(|x'|^2 - \eta_2^2)\right]\right\} \\
&\qquad = c_p'\varepsilon^{-\frac{n}{2}-p} e^{-\frac{\Delta}{\varepsilon}} \sup_{t\geqslant 0}\left\{(t+\eta_2^2)^p e^{-\frac{t}{8\varepsilon}}\right\} \\
&\qquad\qquad \leqslant c_p''\varepsilon^{-\frac{n}{2}-p} e^{-\frac{\Delta}{\varepsilon}} \sup_{t\geqslant 0} t^p e^{-\frac{t}{8\varepsilon}} = c_p\varepsilon^{-\frac{n}{2}} e^{-\frac{\Delta}{\varepsilon}} \to 0.
\end{aligned} \tag{4.34}$$

Here c_p', c_p'', c_p are certain appropriately chosen constants.

Since the order of $\tilde{f} \in S^*$ is finite, then, in view of relation (4.33), the inequality

$$|f_\varepsilon(z)| \leqslant \|f\|_p \|\overline{\psi_\varepsilon(z;\ x')}\|_p$$

holds for some natural number p. Hence it follows from relation (4.34) that for $|z| \leq \eta_1/\sqrt{2}$ (and accordingly also for $z \in K$)

$$|f_\varepsilon(z)| \leqslant \|f\|_p c_p \varepsilon^{-\frac{n}{2}} e^{-\frac{\Delta}{\varepsilon}} \to 0 \quad \text{for} \quad \varepsilon \to +0.$$

The right hand side of this last inequality is independent of z; thus the uniformity of the limiting process has also been established.

The proof for the derivatives does not change in any essential details since

$$\partial_z^\alpha f_\varepsilon(z) = \overline{(\tilde{f}(x'),\ \partial_z^\alpha \psi_\varepsilon(z,\ x'))}.$$

Here

$$\partial_z^\alpha = \frac{\partial^{\|\alpha\|}}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}, \quad \|\alpha\| = \alpha_1 + \ldots + \alpha_n.$$

Thus Lemma 1 is completely proved.

For every $\epsilon > 0$ we now consider the entire functions

$$\tilde{f}_\varepsilon^\pm(z) = \int_{R_n^{(\xi)}} f^\pm(\xi) \exp[iz\xi - \varepsilon|\xi|^2]\, d\xi$$
$$= \overline{(f^\pm(\xi),\ \exp[iz\xi - \varepsilon|\xi|^2])}. \tag{4.35}$$

Next we construct the function of a complex variable ζ

$$\varphi_\varepsilon^\pm(\zeta) = \exp\left[-\frac{1}{\sqrt{1-\zeta^2}}\right] f_\varepsilon^\pm(b\zeta + z(1-\zeta^2)). \tag{4.36}$$

Here $b\zeta + z(1-\zeta^2)$ is a point of the space C^n with coordinates $b_k\zeta + z_k(1-\zeta^2)$, $k = 1, \cdots, n$, and $(b_1, \cdots, b_n)$ is a point of the space R_n which will be mentioned below (see Lemma 4). Evidently the functions $\phi_\epsilon^\pm(\zeta)$ are single-valued, and accordingly holomorphic in the plane of the complex variable ζ with cuts along the straight lines $-\infty < \operatorname{Re}\zeta \leq -1$, $1 \leq \operatorname{Re}\zeta < \infty$, $\operatorname{Im}\zeta = 0$.

REMARK. Here and in what follows we consider the branch of $\sqrt{1-\zeta^2}$ taking on positive values on the segment $-1 < \operatorname{Re}\zeta < 1$, $\operatorname{Im}\zeta = 0$.

LEMMA 2. *The function*

$$\varphi_\varepsilon(\zeta) = \begin{cases} \varphi_\varepsilon^+(\zeta) - T_\varepsilon(\zeta) & \text{for} \quad \operatorname{Im}\zeta > 0, \\ \varphi_\varepsilon^-(\zeta) - T_\varepsilon(\zeta) & \text{for} \quad \operatorname{Im}\zeta < 0, \end{cases} \tag{4.37}$$

where

$$T_\varepsilon(\zeta) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\varphi_\varepsilon^+(t) - \varphi_\varepsilon^-(t)}{t - \zeta}\, dt, \tag{4.38}$$

is holomorphic in the disk $|\zeta| < 1$ *and continuous in the closed disk* $|\zeta| \leq 1$.

PROOF. Indeed, the holomorphy of the function $\phi_\epsilon(\zeta)$ for $\operatorname{Im}\zeta \neq 0$ follows immediately from formulas (4.36)–(4.38). Consider points lying on the segment $-1 < \operatorname{Re}\zeta < 1$, $\operatorname{Im}\zeta = 0$. First of all we note that the function

$$\begin{aligned}\varphi_\varepsilon^+(x) - \varphi_\varepsilon^-(x) = \big[\tilde{f}_\varepsilon^+(bx + z(1-x^2)) \\ - \tilde{f}_\varepsilon^-(bx + z(1-x^2))\big]\exp\left(-\frac{1}{\sqrt{1-x^2}}\right),\end{aligned} \tag{4.39}$$

where $x = \operatorname{Re}\zeta$, belongs to the class $\mathcal{C}^\infty$ on the closed interval $-1 \leq x \leq 1$ and has zeros of infinite order at $x = \pm 1$. Hence it follows [1] that the boundary values $\phi_\epsilon(x \pm i0)$ of the function $\phi_\epsilon(\zeta)$ exist and are continuous, and in view of Sohockiĭ's formula, [2]

$$\begin{aligned}\varphi_\varepsilon(x) = \varphi_\varepsilon(x \pm i0) = \frac{1}{2}\varphi_\varepsilon^+(x) + \frac{1}{2}\varphi_\varepsilon^-(x) \\ - \frac{1}{2\pi i}\ \mathrm{P}\int_{-1}^{1}\frac{\varphi_\varepsilon^+(t) - \varphi_\varepsilon^-(t)}{t-x}\,dt\end{aligned} \tag{4.40}$$

coincide with each other for $|x| < 1$. Therefore the function $\phi_\epsilon(\zeta)$ is holomorphic for $\zeta = x$, $|x| < 1$, and accordingly also in the disk $|\zeta| < 1$. Moreover it can be concluded from formula (4.40) that the function $\phi_\epsilon(x)$ is continuous on the closed interval $-1 \leq x \leq 1$. Thus, by taking account of relations (4.36)–(4.38), we establish that the function $\phi_\epsilon(\zeta)$ is continuous in the whole closed disk $|\zeta| \leq 1$. The lemma is proved.

By the Gauss theorem on arithmetic means we have

$$\varphi_\varepsilon(0) = \frac{1}{2\pi}\int_0^{2\pi}\varphi_\varepsilon(e^{i\theta})\,d\theta. \tag{4.41}$$

By solving relation (4.36) with respect to $f_\epsilon^\pm$ for $\zeta = 0$ and then using equalities (4.37), (4.40) and (4.41), we obtain (on account of our choice of the branch of $\sqrt{1-\zeta^2}$):

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1) See, for example, N. I. Mushelišvili, *Singular integral equations*, Fizmatgiz, Moscow, 1962, Chapter 1; English transl. of 1946 ed., Noordhoff, Groningen, 1953.

2) In this formula the symbol P denotes the Cauchy principal value of the integral.

$$\tilde{f}_\varepsilon^{\pm}(z) = \frac{e}{2\pi}\int_0^{\pi} \tilde{f}_\varepsilon^{+}\left(be^{i\theta} + z(1-e^{2i\theta})\right)\exp\left[-\frac{e^{i\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}}{\sqrt{2\sin\theta}}\right]d\theta$$
$$+\frac{e}{2\pi}\int_{\pi}^{2\pi} \tilde{f}_\varepsilon^{-}\left(be^{i\theta} + z(1-e^{2i\theta})\right)\exp\left[-\frac{e^{-i\left(\frac{\theta}{2}-\frac{3\pi}{4}\right)}}{\sqrt{2|\sin\theta|}}\right]d\theta$$
$$-\frac{e}{2\pi}\int_0^{2\pi} T_\varepsilon(e^{i\theta})\,d\theta + eT_\varepsilon(\pm i0).$$

In the right hand side of these representations we replace $\tilde{f}_\epsilon^{\pm}$ by their expressions (4.35). Then these representations take the form:

$$\tilde{f}_\varepsilon^{\pm}(z) = \frac{e}{2\pi}\int_0^{\pi} \left(\overline{f^{+}(\xi)}, \ \exp\left[ie^{i\theta}b\xi\right.\right.$$
$$\left.\left.+ i(1-e^{2i\theta})z\xi - \varepsilon|\xi|^2\right]\right)\exp\left[-\frac{e^{-i\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}}{\sqrt{2\sin\theta}}\right]d\theta$$
$$+\frac{e}{2\pi}\int_{\pi}^{2\pi}\left(\overline{f^{-}(\xi)}, \ \exp\left[ie^{i\theta}b\xi +\right.\right.$$
$$\left.\left.+ i(1-e^{2i\theta})z\xi - \varepsilon|\xi|^2\right]\right)\exp\left[-\frac{e^{-i\left(\frac{\theta}{2}-\frac{3\pi}{4}\right)}}{\sqrt{2|\sin\theta|}}\right]d\theta$$
$$-\frac{e}{2\pi}\int_0^{2\pi} T_\varepsilon(e^{i\theta})\,d\theta + eT_\varepsilon(\pm i0).$$

Since $f^{\pm}(\xi)$ are continuous functionals over the space S, we can change the order of "integration" in the right hand side of the last expression. Performing this transformation, we obtain the result which is expressed in the form of the following lemma:

LEMMA 3. *For all $z \in C^n$ the following equality holds*:

$$\tilde{f}_\varepsilon^{\pm}(z) = \int_{R_n^{(\xi)}} \overline{f^{+}(\xi)}\, e^{-\varepsilon|\xi|^2} H_1^{+}(z, \xi)\,d\xi$$
$$+\int_{R_n^{(\xi)}} \overline{f^{-}(\xi)}\, e^{-\varepsilon|\xi|^2} H_1^{-}(z, \xi)\,d\xi$$
$$-\frac{e}{2\pi}\int_0^{2\pi} T_\varepsilon(e^{i\theta})\,d\theta + eT_\varepsilon(\pm i0). \qquad (4.42)$$

Here

$$H_1^{\pm}(z, \xi) = \frac{e}{2\pi}\int_0^{\pi} \exp\left[ie^{\pm i\theta}b\xi + i(1 - e^{2i\theta})z\xi - \frac{e^{-i\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}}{\sqrt{2\sin\theta}}\right] d\theta. \tag{4.43}$$

3. Completion of the proof of Theorems 19.1 and 19.2. Our aim is to pass to the limit as $\epsilon \to +0$ in both sides of relation (4.42). In the sequel we take a number $\eta_1 < \eta$ such that the domain $D \subset E_{\eta_1}$. Here

$$E_{\eta_1} = \left\{ |z_1| + \sqrt{|z_2|^2 + \dots + |z_n|^2} > \frac{\eta_1}{2\sqrt{2}} \right\},$$

and D is the domain specified in the formulation of Theorem 19.2. We shall prove the following lemma:

LEMMA 4. *If* $b = (\eta_2/\sqrt{2}, 0, \cdots, 0)$, *where* $\eta_1 < \eta_2 < \eta$, *then the function* $T_\epsilon(\zeta)$ *for* $\epsilon \to +0$ *tends uniformly towards zero for* $(z, \zeta) \in \overline{E}_{\eta_1} \times \{|\zeta| \le 1\}$.

PROOF. By equalities (4.38) and (4.39) we have

$$T_\varepsilon(\zeta) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\Phi_\varepsilon(z, t)}{t-\zeta} \exp\left(-\frac{1}{\sqrt{1-t^2}}\right) dt, \tag{4.44}$$

where

$$\Phi_\varepsilon(z, t) = \tilde{f}_\varepsilon^{+}(bt + z(1-t^2)) - \tilde{f}_\varepsilon^{-}(bt + z(1-t^2)). \tag{4.45}$$

Hence it follows that for $(z, \zeta) \in \overline{E}_{\eta_1} \times \{|\zeta| \le 1\}$

$$|T_\varepsilon(\zeta)| \leqslant c \max_{(z, t)\in \overline{E}_{\eta_1} \times \{|t| \leqslant 1\}} \left[|\Phi_\varepsilon(z, t)| + \left|\frac{\partial}{\partial t}\Phi_\varepsilon(z, t)\right|\right], \tag{4.46}$$

where the quantity c is independent of the position of the point (z, ζ) and the choice of the number ϵ. Since $\tilde{f}(x) = \tilde{f}^{+}(x) - \tilde{f}^{-}(x) = 0$ for $|x| < \eta$, we can apply Lemma 1 to the function $\Phi_\epsilon(z, t)$. To this end, in view of relation (4.45), it is only necessary to establish that the points $z' = bt + z(1 - t^2)$, where $z \in \overline{E}_{\eta_1}$, $|t| \le 1$, satisfy the condition $|z'| \le \eta_3/\sqrt{2}$ for some $\eta_3 < \eta$.

We take $\eta_3 = \eta_2$. Since for $z \in \overline{E}_{\eta_1}$

$$|x_1| \leqslant |z_1| \leqslant \frac{\eta_1}{2\sqrt{2}} < \frac{\eta_2}{2\sqrt{2}},$$

we have

$$|z|^2 (1 - t^2)^2 + \frac{\eta_2^2}{2} t^2 + \sqrt{2}\eta_2 t (1 - t^2) x_1 \leqslant \frac{\eta_2^2}{2}$$

for all $(z, t) \in \overline{E}_{\eta_1} \times \{|t| \le 1\}$. Thus the condition for the applicability of Lemma 1 is satisfied, since we may conclude in view of formula (4.45) that the limits $\Phi_\epsilon(z, t) \to 0$ and $\partial\Phi_\epsilon(z, t)/\partial t \to 0$ for $\epsilon \to +0$ are attained uniformly for all $(z, t) \in \overline{E}_{\eta_1} \times \{|t| \le 1\}$. Hence, by using the estimate (4.46), we obtain our assertion. Lemma 4 is proved.

Let $\delta(t)$ be a function of the class $\mathcal{C}^\infty$, equal to zero for $t \le -2$ and unity for $t \ge -1$. By means of this function we form the entire function

$$H_2^{\pm}(z, \xi) = \frac{e}{2\pi} \delta(\pm \xi_1) \delta(\xi_1^2 - \xi_2^2 - \dots - \xi_n^2) \int_0^\pi \exp\Bigg[ie^{\pm i\theta} b\xi + i(1 - e^{2i\theta}) z\xi - \frac{e^{-i\left(\frac{\theta}{2} - \frac{\pi}{4}\right)}}{\sqrt{2 \sin\theta}} \Bigg] d\theta. \tag{4.47}$$

Taking into account the fact that $f^{\pm}(\xi) = 0$ outside the cones $\Gamma^{\pm}$, respectively, and using equations (4.43) and (4.47), we may replace the representations (4.42) by the following formula:

$$\tilde{f}_\varepsilon^{\pm}(z) = \int_{R_n^{(\xi)}} \overline{f^{+}(\xi)}\, e^{-\varepsilon|\xi|^2} H_2^{+}(z, \xi)\, d\xi + \int_{R_n^{(\xi)}} \overline{f^{-}(\xi)}\, e^{-\varepsilon|\xi|^2} H_2^{-}(z, \xi)\, d\xi - \frac{e}{2\pi} \int_0^{2\pi} T_\varepsilon(e^{i\theta})\, d\theta + eT_\varepsilon(\pm i0). \tag{4.48}$$

We wish to realize the passage to the limit for $\epsilon \to +0$ in the representations (4.48), uniformly for the points $z \in \overline{E}_{\eta_1}$. In the topology assumed for the space S (the passage to the limit is realized in this sense; see subsection 6 of the Introduction), this is possible if the norms of all the derivatives $\partial_z^\alpha H_2^{\pm}(z, \xi)$ are bounded by identical numbers for all points $z \in \overline{E}_{\eta_1}$.

Starting from equality (4.47), we obtain

$$\left\|\partial_z^\alpha H_2^\pm(z,\xi)\right\|_p \leqslant c'_{p,\alpha} \sup_{\xi\in G^\pm,\ 0\leqslant\theta\leqslant\pi} |\xi|^{2p+\|\alpha\|}\left|\exp\left[ie^{\pm i\theta}b\xi + \right.\right.$$
$$\left.\left. + i(1-e^{2i\theta})z\xi - \frac{e^{-i\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}}{\sqrt{2\sin\theta}}\right]\right|, \tag{4.49}$$

where $c'_{p,\alpha}$ are some constants, $G^\pm = \{\pm\xi_1 \geq R,\ \xi_1^2 - \xi_2^2 - \cdots - \xi_n^2 > -2\}$, while the number $R > 0$ is so chosen that

$$\Delta = \frac{1}{\sqrt{2}}\left(\eta_2 - \sqrt{1+\frac{2}{R^2}\eta_1}\right) > 0, \qquad R \geqslant \frac{1}{4\Delta}.$$

Such a choice of the number R is always possible since $\eta_2 > \eta_1$.

Let a point $(z, \xi, \theta) \in E_{\eta_1} \times G^\pm \times \{0 \leq \theta \leq \pi\}$. Then

$$\operatorname{Re}\left[ie^{\pm i\theta}b\xi + i(1-e^{2i\theta})z\xi - \frac{e^{-i\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}}{\sqrt{2\sin\theta}}\right]$$
$$= \mp b\xi\sin\theta + x\xi\sin 2\theta - (1-\cos 2\theta)y\xi - \frac{\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}{\sqrt{2\sin\theta}}$$
$$\leqslant \mp\sin\theta\left[\frac{\eta_2}{\sqrt{2}}\xi_1 \mp 2(x\xi\cos\theta - y\xi\sin\theta)\right] - \frac{1}{2\sqrt{\sin\theta}}$$
$$\leqslant \mp\sin\theta\left[\frac{\eta_2}{\sqrt{2}}\xi_1 \mp 2(|\xi_1||z_1| + \right.$$
$$\left. + \sqrt{\xi_2^2+\cdots+\xi_n^2}\sqrt{|z_2|^2+\cdots+|z_n|^2})\right] - \frac{1}{2\sqrt{\sin\theta}}$$
$$\leqslant \mp\xi_1\sin\theta\left(\frac{\eta_2}{\sqrt{2}} - 2|z_1| - \right.$$
$$\left. - 2\sqrt{1+2\xi_1^{-2}}\sqrt{|z_2|^2+\cdots+|z_n|^2}\right) - \frac{1}{2\sqrt{\sin\theta}}$$
$$\leqslant \mp\sin\theta\,\frac{\xi_1}{\sqrt{2}}\left(\eta_2 - \sqrt{1+2R^{-2}\eta_1}\right) - \frac{1}{2\sqrt{\sin\theta}}$$
$$= \mp\Delta\xi_1\sin\theta - \frac{1}{2\sqrt{\sin\theta}}.$$

Using the estimate obtained, we also find from inequality (4.49) that

$$\left\|\partial_z^\alpha H_2^\pm(z,\xi)\right\|_p$$
$$\leqslant c''_{p,\alpha} \sup_{\pm\xi > R,\ 0\leqslant\theta\leqslant\pi} |\xi_1|^\beta \exp\left[\mp\Delta\xi_1\sin\theta - \frac{1}{2\sqrt{\sin\theta}}\right], \tag{4.50}$$

where β, $c''_{p,\alpha}$ are some constants. Moreover, we find that for $x \geq R \geq 1/4\Delta$

$$\sup_{0\leqslant\theta\leqslant\pi}\left[-\Delta x \sin\theta - \frac{1}{2\sqrt{\sin\theta}}\right] = -3\left(\frac{\Delta x}{16}\right)^{\frac{1}{3}}. \tag{4.51}$$

From relations (4.50) and (4.51) we obtain

$$\|\partial_z^\alpha H_2^\pm(z, \xi)\|_p \leqslant c''_{p,\alpha} \sup_{|\xi_1|\geqslant R} |\xi_1|^\beta \exp\left[-3\left(\frac{\Delta|\xi_1|}{16}\right)^{\frac{1}{3}}\right] \leqslant c_{p,\alpha} < \infty.$$

Thus the uniform boundedness of the norms for $z \in \bar{E}_{\eta_1}$ is established. Passing to the limit for $\epsilon \to +0$ in the right hand side of the representations (4.48), we find that

$$\lim_{\varepsilon\to+0} \widetilde{f_\varepsilon^\pm}(z) = \int_{R_n^{(\xi)}} \overline{f^+(\xi)}\, H_2^+(z, \xi)\, d\xi + \int_{R_n^{(\xi)}} \overline{f^-(\xi)}\, H_2^-(z, \xi)\, d\xi = f(z). \tag{4.52}$$

In the right hand side of equality (4.52) we obtain one and the same function $f(z)$ holomorphic for $z \in \bar{E}_{\eta_1}$. On the other hand, in view of equality (4.35) for $z_k = x_k$ $(k = 1, \cdots, n)$

$$\lim_{\varepsilon\to+0} \widetilde{f_\varepsilon^\pm}(x) = \widetilde{f^\pm}(x) \tag{4.53}$$

(where the limit is taken in the sense of the topology of the space S^*). From the limiting relations (4.52) and (4.53) we conclude that

$$\tilde{f}(x) = \widetilde{f^+}(x) = \widetilde{f^-}(x) \quad \text{for} \quad x \in E_{\eta_1}.$$

This means that the function $f(z)$ that we have constructed is indeed an analytic continuation of the generalized function $\tilde{f}^\pm(x)$.

Passing from the representation (4.52) to its Fourier transform (see the corresponding formulas in subsection 6 of the Introduction), we obtain the representation (4.30). In this connection the functions $H^\pm(z, x') = (2\pi)^{-n} H_2^\pm(z, \xi)$ will have all the properties indicated in Theorem 19.2.

We still note that the function $f(z)$ that we have constructed is holomorphic in the whole domain E_η, since the number η_1 may be taken as close as we like to the number η.

Thus the proof of Theorems 19.2 and 19.1 is complete.

REMARK 1. The condition $V^+ \cap (-V^-) \neq \emptyset$ may be weakened and replaced by the condition $O(V^+) \cap O(V^-) \neq \emptyset$; Theorem 19.1 remains valid in this case since the estimate (4.27) is also preserved in T^+ and T^- (possible, with other coefficients A, $\rho_\pm$, $m_\pm$; see Vladimirov [4]).

2. As was proved by Epstein [1], the condition 2) of Theorem 19.1 may be dropped from its assumptions. The consequent strengthening of the "edge of the wedge" theorem establishes the possibility of extending the open set $T^+ \cup T^-$ with respect to the set of holomorphic functions satisfying only the condition (4.28).

4. Consequences of the "edge of the wedge" theorem. Assume that functions $f^\pm(z)$ satisfy the conditions 1), 2) and 3) of Theorem 19.1. Set $V = O(V^+) \cap O(-V^-)$ (we apply Theorem 19.1 on account of Remark 1).

A line of the class $\mathcal{C}^1$ in the space R_n is said to be *V-time-like* if all of its tangent vectors belong to the cone $V \cup (-V)$.

The name *V-convex hull* of an open set $G \subset R_n$ is applied to the smallest open set $B_V(G) \supset G$ that has the following property: if points $x_1, x_2 \in B_V(G)$ can be joined by a V-time-like straight line segment lying entirely in $B_V(G)$, then all V-time-like lines joining the points x_1 and x_2 are contained in the open set $B_V(G)$.

Assume that for the open set G (see the condition 3) of Theorem 19.1) the condition 4) is also satisfied, and that $B_V(G) = R_n$. This last equality will hold, for example, if $V = \Gamma^+ = -\Gamma^-$, $G = \{x_2^2 + \cdots + x_n^2 < \epsilon\}$, where ϵ is an arbitrary positive number. Then it is easy to see that $B_V(G) = R_n^{(\mathrm{Re})}$.

It can be proved that under these conditions (see Vladimirov [2]): *The functions $f^\pm(z) = P(z)$, where $P(z)$ is a polynomial of degree not larger than* $\min(m_+, m_-)$.

Theorem 19.1 does not settle the limits of possible extension of the domain $T^+ \cup T^-$ with respect to the collection of holomorphic functions considered there. For a wide family of functions such limits were found, however, in a paper by Vladimirov (see Vladimirov [1], Theorem 7). [1)]

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1) Further applications of the "edge of the wedge" theorem are to be found in V. S. Vladimirov, *Methods of the theory of functions of several complex variables*, Izdat. "Nauka", Moscow, 1964 (Russian). This book contains many other methods of holomorphic extension.

CHAPTER V

BIHOLOMORPHIC MAPPINGS

The first part of the present chapter is devoted to the metric invariant under biholomorphic mappings and to its applications: namely, the investigation of domains having continuous groups of automorphisms, to methods for obtaining estimates for biholomorphic mappings and to a number of other problems. In the second part a survey is given of some results of the theory of holomorphic mappings that are closely connected with the foregoing, but are obtained without using an invariant metric.

§20. SETS OF HOLOMORPHIC MAPPINGS

The present section is preliminary to the subsequent exposition.

Suppose that in a domain D over the space C^n_z defined by coverings $\{S_i, \epsilon_{ij}\}$ we are given the sequence of systems of holomorphic functions

$$(T_\nu) \qquad w_k = w_k^\nu(z), \quad k = 1, \cdots, n; \ \nu = 1, 2, \cdots. \tag{5.1}$$

Here each of the functions $w_k^\nu(z)$ is given as a collection of holomorphic functional elements $[w_k^\nu(z)]_i$, $z \in S_i$. As was established above, if the Jacobian $\partial w^\nu/\partial z = \partial[w^\nu]_i/\partial z$ (where the index i takes values indicating the number of the elementary domain S_i) does not vanish in the domain D, the system of functions (T_ν) defines a biholomorphic mapping onto a domain D^*_ν over the space C^n_w (see Theorem 10.1, (I)) without interior branch points. If the Jacobian $\partial w^\nu/\partial z$ vanishes in the domain D, but Osgood's condition is satisfied at all points of the domain D where $\partial w^\nu/\partial z = 0$, then the system of functions (T_ν) defines a generalized biholomorphic mapping of the domain D onto an interiorly-branching domain D^*_ν over the space C^n_w (see §10.4, Chapter II, (I)).

Assume that the limits $\lim_{\nu\to\infty} w_k^\nu(z) = w_k(z)$, $k = 1, \cdots, n$, exist and are uniformly attained in the domain D. Our aim is to make clear the properties of the system of limit functions

$$(T) \qquad w_k = w_k(z), \quad k = 1, \cdots, n. \tag{5.2}$$

Evidently they are holomorphic in the domain D.

1. Local properties of limit mappings.

THEOREM 20.1 (Carathéodory [3]). *The sequence of the systems of holomorphic functions (T_ν) converges uniformly to the system of holomorphic functions (T) in a neighborhood of a point $z^0 \in C^n_z$. If the Jacobian $(\partial w/\partial z)_{z^0} \neq 0$ for the system of functions (T), then we can find neighborhoods $U_{z^0} \subset C^n_z$, $U^*_{w^0} \subset C^n_w$ (where $w^0 = Tz^0$) and a natural number ν_0, such that the systems of functions (T_ν) for $\nu \geq \nu_0$ will biholomorphically map the neighborhood U_{z^0} onto a domain of the space C^n_w containing the neighborhood $U^*_{w^0}$.*

PROOF. In view of Theorem 7.1, (I) the system of functions (5.2) defines a biholomorphic mapping of some polycylinder $S(z^0, r)$ onto a certain neighborhood $U^*_{w^0}$ of a point w^0. We choose the polycylinder $S(z^0, r)$ in such a way that the minimum of the modulus of the Jacobian for the system of functions (T) in that polycylinder will differ from zero. Then because of the uniform convergence of the sequence $\{T_\nu\}$, there exists a natural number μ such that for $\nu \geq \mu$ the Jacobian of the system of functions (T_ν) will also be different from zero. Hence, in view of Theorem 10.1, (I), it follows that these systems of functions (T_ν) map the polycylinder $S(z^0, r)$ biholomorphically onto some, generally speaking, multiple-sheeted domains over the space C^n_w. We shall show that the radius r of the polycylinder $S(z^0, r)$ may be chosen so small that all of these domains will be single-sheeted.

Indeed, otherwise there would exist a sequence of numbers ν_l, where $l = 1, 2, \cdots$, $\nu_l < \nu_{l+1}$, and two sequences of points $\{z'^l\}$, $\{z''^l\}$ belonging to the polycylinder $S(z^0, r)$ such that $\lim_{l\to\infty} z'^l = \lim_{l\to\infty} z''^l = z^0$, $z'^l \neq z''^l$, but $T_{\nu_l} z'^l = T_{\nu_l} z''^l$. We set $\alpha^l_k = (z''^l_k - z'^l_k) r_l^{-1}$, $\phi^l_k(t) = w^{\nu_l}_k (z'^l + \alpha^l t) - w^{\nu_l}_k (z'^l)$. Here r_l is the distance between the points z'^l and z''^l, and t is a complex parameter taking values for which the point $(z'^l + \alpha^l t)$ with coordinates $z'^l_k + \alpha^l_k t$, $k = 1, \cdots \cdots, n$, belongs to the polycylinder $S(z^0, r)$. We may always suppose (if necessary, replacing the sequence (T_{ν_l}) by a subsequence) that the limits $\lim_{l\to\infty} \alpha^l_k = \alpha_k$ exist.

Then the functions $\phi^l_k(t)$ converge uniformly as $l \longrightarrow \infty$, in some neighborhood of the value $t = 0$, to the functions

$$\phi_k(t) = w_k(z^0 + \alpha t) - w_k(z^0). \tag{5.3}$$

Evidently $\phi_k^l(r_l) = \phi_k^l(0) = 0$. Therefore

$$\varphi_k'(0) = \lim_{l\to\infty} \frac{\varphi_k^l(r_l) - \varphi_k^l(0)}{r_l} = 0.$$

By (5.3) this means that

$$\sum_{s=1}^{n} \alpha_s \left(\frac{\partial w_k}{\partial z_s}\right)_{z^0} = 0, \qquad k = 1, \ldots, n. \tag{5.4}$$

On the other hand, since $\sum_{s=1}^{n} |\alpha_s|^2 = 1$, all the α_s cannot vanish simultaneously. Hence it follows that the determinant of the homogeneous system of equations (5.4), i.e., $(\partial w/\partial z)_{z^0}$, is equal to zero, which contradicts the assumption of the theorem to be proved. Thus it has been established that our supposition is invalid.

From the foregoing it follows that the distances d_0 of the point w^0 to the set $T\partial S(z^0, r) = \partial T S(z^0, r)$ and d_ν of the points $w^\nu = T_\nu z^0$ to the sets $T_\nu \partial S(z^0, r) = \partial T_\nu S(z^0, r)$ for $\nu \geq \mu$ are distinct from zero. Because of the uniform convergence of the sequence $\{T_\nu\}$

$$\lim_{\nu\to\infty} d_\nu = d_0, \qquad \lim_{\nu\to\infty} w^\nu = w^0. \tag{5.5}$$

Therefore the domain $T_\nu S(z^0, r)$ contains the polycylinder $S(w^\nu, d_\nu/\sqrt{n})$. Then, in view of the limiting equalities (5.5), there exists a number ν_0 such that for $\nu \geq \nu_0$ the domain $T_\nu S(z^0, r)$ over the space C_w^n contains the polycylinder $S(w^0, d_0/2\sqrt{n})$. Theorem 20.1 is proved.

Theorem 20.1 can be also extended to infinitely distant points. In that case one must use the modification of the Jacobian for the system of functions described in §10.3, Chapter II, (I).

2. Global properties of limit mappings. First of all we shall prove one auxiliary proposition.

Lemma 1. *In a domain $D \subset C^n$ a sequence $\{f_\nu(z)\}$ consisting of holomorphic and nonvanishing functions converges uniformly to a function $f(z)$. Then this last function is either identically equal to zero or does not vanish in the domain D.*

Proof. Let $n = 1$. If the limit function is constant, our assertion is evident. If $f(z) \not\equiv$ const, one can find for any point $z^0 \in D$ a circle $E = \{|z - z^0| = r\} \subset D$ on which $f(z) \neq 0$. Then there exists a number $\epsilon > 0$ such that $|f(z)| > \epsilon$ and $|f_\nu(z)| \geq \epsilon/2$ when $z \in E$ and $\nu > \nu_0$. Here ν_0 is some natural number chosen appropriately. Since $f_\nu(z) \neq 0$ for $z \in D$, we also have $|f_\nu(z)| \geq \epsilon/2$ for $|z - z^0| \leq r$.

Hence it follows that $f(z^0) \neq 0$.

Let $n > 1$, the point $z^0 \in D$ and the function $f(z) \not\equiv$ const in the domain $D \subset C^n$. Then one can choose numbers $\alpha_1, \cdots, \alpha_n$ in such a way that the function $\phi(t) = f(z^0 + \alpha t)$ is not constant in a neighborhood of the point $t = 0$. Here t is a complex parameter and $z^0 + \alpha t$ is the point with coordinates $z_k^0 + \alpha_k t$, $k = 1, \cdots, n$. We set $\phi_\nu(t) = f_\nu(z^0 + \alpha t)$; then in some neighborhood of the point $t = 0$ all of the functions $\phi_\nu(t) \neq 0$ and $\lim_{\nu\to\infty} \phi_\nu(t) = \phi(t)$ is reached uniformly. Hence, as we have seen, it follows that $\phi(0) = f(z^0) \neq 0$. The lemma is proved.

Now we can obtain the

THEOREM 20.2 (H. Cartan [2]). *Suppose that each of the functions (T_ν) defines a biholomorphic mapping of a domain D over the space C_z^n onto a domain D_ν^* over the space C_w^n without interior branch points, and that the systems of functions (T_ν) converge uniformly in the domain D, as $\nu \to \infty$, to the system of functions (T). Then the limit system of functions (T) either defines a degenerate mapping or a biholomorphic mapping of the domain D onto some domain over the space C_w^n without interior branch points.*

If a domain $D \subset C_z^n$ and each system of functions (T_ν) defines a biholomorphic mapping of the domain D onto a domain $D_\nu^* \subset C_w^n$, then the limit system of functions (T) either defines a degenerate mapping or a biholomorphic mapping of the domain D onto some domain of the space C_w^n.

PROOF. This theorem follows almost immediately from Lemma 1. If $n = 1$, the system of functions (T_ν) is reduced to one function $w = w^\nu(z)$, and the system of functions (T) to a function $w = w(z)$, and moreover in view of our assumptions $[w^\nu(z)]' \neq 0$ at all points $z \in D$ (see the end of §10.1, Chapter II, (I)). Then, by Lemma 1, either $w'(z) \neq 0$ for $z \in D$ or $w(z) \equiv$ const. In the first case the function $w = w(z)$ realizes a biholomorphic (conformal) mapping of the domain D onto some domain D^* over the w-plane (see Theorem 10.1, (I)). If the domain $D \subset C_z^1$ and (T_ν) is its single-sheeted mapping, then the difference $f_\nu(z'') - f_\nu(z')$, where $z', z'' \in D$, $z' - z'' \neq 0$, does not vanish. Then, by Lemma 1, the difference $f(z'') - f(z')$ is constant (evidently it must be zero) or different from zero everywhere in the domain D.

Consider now the case $n > 1$. In view of Theorem 7.3, (I) (see also the end of §10.1, Chapter II, (I)), for any index ν the Jacobian of the system of functions (T_ν) is either identically equal to zero or vanishes nowhere in the domain D. In the first case the system of functions (T) defines a degenerate mapping, while in

the second case (see Theorem 10.1, (I)) it defines a biholomorphic mapping of the domain D.

It now remains for us to prove the last assertion. Let the domain $D \subset C^n_z$ and let the system of functions (T_ν) for any index ν define a biholomorphic mapping of it. We must show that if the system of functions (T) defines a nondegenerate mapping, then for any two points $z', z'' \in D$, $z' \neq z''$, we have also $w' \neq w''$, where $w' = Tz'$, $w'' = Tz''$. We may always assume that $z''_k = w''_k = 0$, $k = 1, \cdots, n$. After this normalization it suffices to establish that for $|z'_1|^2 + \cdots + |z'_n|^2 > 0$ we have also $|w'_1|^2 + \cdots + |w'_n|^2 > 0$. The Jacobian of the system of functions (T) differs from zero at the origin of coordinates. Therefore this system defines a biholomorphic mapping of some neighborhood $\{|z_1|^2 + \cdots + |z_n|^2 < \rho\}$ of the origin of coordinates. By Theorem 20.1 the number $\rho > 0$ may be taken so small that for $\nu \geq \nu_0$ all the systems (T_ν) define biholomorphic mappings of this neighborhood onto domains containing the hyperball $\{|w_1|^2 + \cdots + |w_n|^2 < \sigma\}$ (where the choice of the quantity $\sigma > 0$ depends on ρ). Then all of these mappings carry points for which $|z_1|^2 + \cdots + |z_n|^2 > \rho$ into points for which $|w_1|^2 + \cdots + |w_n|^2 \geq \sigma$.

This property is preserved under passage to the limit, so that our assertion is confirmed.

The following proposition, converse to Theorem 20.2 in a certain sense will now be shown to be true.

THEOREM 20.3 (H. Cartan [2]). *In some domain D over the space C^n having no interior branch points a sequence of systems of holomorphic functions (T_ν), $\nu = 1, 2, \cdots$, converges uniformly to a system of functions (T) which defines a biholomorphic mapping of that domain D. Then to each subdomain $D_0 \ll D$ there corresponds a number ν_0 such that for $\nu \geq \nu_0$:* 1) *the system (T_ν) defines a biholomorphic mapping of the domain D_0;* 2) *the domain $T_\nu D_0$ is a subdomain of the domain $D^* = TD$ and $T_\nu D_0 \ll D^*$;* 3) *if a domain $D_1 \ll D_0$ is a subdomain of the domain D, then the domain TD_1 is a subdomain of the domain $T_\nu D_0$.*

PROOF. The first assertion of the theorem follows from the fact that $\lim_{\nu\to\infty} \partial w^\nu/\partial z = \partial w/\partial z$; this limit is reached uniformly in the domain D where $\partial w/\partial z \neq 0$. Therefore for sufficiently large values of ν, for example for $\nu \geq \nu_1$, we have $\partial w^\nu/\partial z \neq 0$ for $z \in D_0$ and, accordingly, by Theorem 10.1, (I), the system of functions (T_ν) defines a biholomorphic mapping of the domain D_0.

We turn to the verification of the second assertion of the theorem. Evidently the domain TD_0 is a subdomain of the domain D^* and $TD_0 \Subset D^*$. It is known that

$\lim_{\nu\to\infty}(T_\nu) = (T)$ is reached uniformly in the domain D and that $\lim_{\nu\to\infty} T_\nu D_0 = TD_0$. Thus for sufficiently large values of ν (for $\nu \geq \nu_2 \geq \nu_1$) it follows that $T_\nu D_0 \ll D^*$. We must prove that for $\nu \geq \nu_3 \geq \nu_2$ the domain $T_\nu D_0$ is a subdomain of the domain D^*. Suppose that such a number ν_3 does not exist. Then there must exist a sequence of numbers ν_l, $l = 1, 2, \cdots$, and points $\mathfrak{z}^l, \zeta^l \in D_0$ (where $\mathfrak{z}^l \neq \zeta^l$, $\lim_{l\to\infty} \mathfrak{z}^l = \mathfrak{z}$, $\lim_{l\to\infty} \zeta^l = \zeta$, the points $\mathfrak{z}, \zeta \in D$) such that to points $T_{\nu_l}\mathfrak{z}^l$, $T_{\nu_l}\zeta^l \in T_{\nu_l}D_0$ there correspond (in view of the correspondence $T_{\nu_l}D_0 \ll D^*$) coincident points of the domain D^*:

$$(T_{\nu_l}\mathfrak{z}^l)^* = (T_{\nu_l}\zeta^l)^*. \tag{5.6}$$

The points $(T_{\nu_l}\mathfrak{z}^l)^*$ and $(T_{\nu_l}\zeta^l)^*$ have the same coordinates as the points $T_{\nu_l}\mathfrak{z}^l$ and $T_{\nu_l}\zeta^l$. These coordinates are the values at the points $\mathfrak{z}^l$ and ζ^l of the functions constituting the system (T_{ν_l}).

Passing to the limit in equality (5.6) as $l \longrightarrow \infty$, we find that $T\mathfrak{z} = T\zeta$, and accordingly, as long as T is a biholomorphic mapping of the domain D, it may be concluded that $\mathfrak{z} = \zeta$. Hence in view of Theorem 20.1 it follows that $\underline{\mathfrak{z}}^l = \underline{\zeta}^l$. But, from the coincidence of the geometrical points, there also follows the coincidence of the corresponding analytical points $\mathfrak{z}_l$ and ζ_l (since the point $\mathfrak{z}$ is not an interior branch point of the domain D, the latter having in general no such points). The equality $\mathfrak{z}_l = \zeta_l$ contradicts our original hypotheses and we must reject the supposition we have made.

Thus the second assertion of the theorem is proved. The proof of the third assertion is similar to that of the second; we shall not discuss it.

REMARK. From the theorem just proved it follows (in view of its second assertion) that if the domain D^* is single-sheeted, then for sufficiently large values of the index ν the domains $T_\nu D_0$ are also single-sheeted.

3. Sequences of automorphisms.

THEOREM 20.4 (H. Cartan [1]). *If a sequence of automorphisms T_ν of a bounded domain D over the space C^n converges uniformly in this domain to a mapping T and there exists at least one pair of points $\mathfrak{z}, \zeta \in D$ such that $T\mathfrak{z} = \zeta$, then T is an automorphism of the domain D.*

REMARK. In this theorem we consider interiorly branching domains. Their automorphisms are generalized biholomorphic mappings of these domains onto themselves. When domains having no interior branch points are considered, these automorphisms are biholomorphic mappings of the corresponding domains onto

themselves.

The proof of Theorem 20.4 is divided into the derivation of a series of lemmas.

LEMMA 2. *Let D and D^* be domains over the space C^n and let T_ν (where $\nu = 1, 2, \cdots$) be a sequence of holomorphic mappings of the domain D into the domain D^*. If the mappings T_ν converge uniformly in the domain D to a nondegenerate mapping T, then there exist single-sheeted subdomains E and E^* of the domains D and D^*, respectively, such that $TE = E^*$.*

PROOF. The points Tz, where the points $z \in D$, are limiting points of the domain D^*. Since the mapping T is nondegenerate, they cannot all belong to the boundary ∂D^* of the domain D^*. Let $\zeta \in D$ be a point such that the point $\zeta^* = T\zeta \in D^*$. Then $TU_\zeta \subset D^*$, where $U_\zeta \subset D$, is also some neighborhood of the point ζ. Among the points of this neighborhood one can find a point $\mathfrak{z}$ such that neither it nor the point $T\mathfrak{z} = \mathfrak{z}^*$ is a branch point of the corresponding domains. Then the Jacobian of the system of functions (T) defining the mapping T differs from zero at the point $\mathfrak{z}$; the mapping T transforms some neighborhood of the point $\mathfrak{z}$ into some neighborhood of the point $\mathfrak{z}^*$. With this the lemma is proved.

LEMMA 3. *Let D and D^* be domains over the space C^n, and let $T: D \to D^*$ and $T^*: D^* \to D$ be holomorphic mappings. If the superposition $T^* \circ T$ represents an automorphism of the domain D, then T and T^* are generalized biholomorphic mappings of the corresponding domains, and moreover $TD = D^*$ and $T^*D^* = D$.*

PROOF. Under our conditions T is a generalized biholomorphic mapping of the domain D onto some subdomain of the domain D^*. For if points $z_1, z_2 \in D$ and $z_1 \neq z_2$, then we also have $Tz_1 \neq Tz_2$; otherwise the equality $T^*(Tz_1) = T^*(Tz_2)$ would hold, which is impossible since $T^* \circ T$ is an automorphism of the domain D. It can also be established that T^* is a generalized biholomorphic mapping of the domain D^* into the domain D. Thus the mappings T and T^* have no exceptional sets in the domains D and D^*, respectively. Let $D_1^* = TD$. Since the mapping $T^* \circ T$ is an automorphism of the domain D, we have $T^*D_1^* = D$; hence it follows that the domain D_1 has no boundary points belonging to the domain D^* (under the operation of the mapping T^* the boundary points must also be carried onto interior points of the domain D). Thus D_1^* is a subdomain of the domain D which has no boundary points belonging to the domain D^*. Hence it follows that $D_1^* = D$. Lemma 3 is proved.

COROLLARY. *If the superposition of two holomorphic mappings of a domain D into itself represents an automorphism, then each of these mappings is an auto-*

morphism of the domain D.

We now turn to the direct proof of Theorem 20.4. First of all we shall show that T is a nondegenerate mapping.

Evidently there exists, in view of the assumptions of Theorem 20.4, a neighborhood $U_{\mathfrak{z}} \Subset D$ of the point $\mathfrak{z}$ such that for a point $z' \in U_{\mathfrak{z}}$

$$\lim_{\nu \to \infty} T_\nu z' = Tz' = z'^* \in D.$$

We may always assume that neither the point z' nor the point z'^* is a branch point of the domain D. Our assertion will be proved if we show that $|\partial w^\nu / \partial z|_{z'} > \alpha > 0$ for any number ν. However, this inequality holds since the functions defining the inverse mappings T_ν^{-1} are bounded everywhere in the domain D (since these define mappings of the domain D onto the bounded domain D). By considering these functions in some polycylinder $S(z', \delta) \Subset D$, where δ is sufficiently small, and applying Cauchy's integral, we find that the partial derivatives of these functions are also bounded. Hence the required result follows.

We note that from our reasoning it follows that if the mapping T is degenerate, it transforms the domain D into some set belonging to its boundary.

For the completion of the proof of Theorem 20.4 it now suffices to show that the following lemma holds:

LEMMA 4. *If a sequence of automorphisms T_ν, $\nu = 1, 2, \cdots$, of a bounded domain D converges uniformly to a nondegenerate mapping T, then this mapping T is itself an automorphism of the domain D.*

PROOF. In view of the foregoing there exist two points z', $z'^* \in D$ (not being branch points of the domain) and their small neighborhoods $V \Subset D$ and $V^* \Subset D$, such that $z'^* = Tz$, $V^* = TV$. The first assertion of Theorem 20.3 implies that for $\nu \geq \nu_0(V)$ the mappings T_ν^{-1} are biholomorphic in the domain V^*. In view of the second and third assertions of that theorem the sequence of the mappings T_ν^{-1}, $\nu = 1, 2, \cdots$, converges uniformly in the domain V^* to a mapping T^{-1}. On account of the boundedness of the domain D (for each index k; see formula (5.1)), the systems of functions defining the mappings T_ν^{-1} form normal families; therefore from the uniform convergence of the sequence of the mappings T_ν^{-1} to the mapping T^{-1} in the domain $V^* \Subset D$ it follows that the sequence converges uniformly in the whole domain D. The limit mapping thus obtained coincides with the mapping T^{-1} in the domain V^* and therefore does not differ from it in the whole domain D.

If T and T^{-1} are mappings of the domain D into itself, then in view of the corollary of Lemma 3 ($T^{-1} \circ T$ is the identity mapping) the mapping T is an

automorphism of the domain D. Suppose that T is not a mapping into the domain D. Then there exists a line $L \subset D$ (its end points being included) joining the point z' to the point z'', such that $(TL \setminus Tz'') \subset D$ and the point $Tz'' \in \partial D$. We consider the sequence of the mappings $T_\nu^{-1}\, T_\mu$ (where ν is fixed and $\mu = 1, 2, \cdots$). The systems of functions defining them form normal families; in the domain $V \Subset D$ this sequence converges uniformly. Therefore the uniform convergence of the above sequence holds in the whole domain D. Let $\lim_{\mu\to\infty} T_\nu^{-1}\, T_\mu = T_\nu^{-1}\, T = \tau_\nu$. Then the points $\tau_\nu z$, where $z \in L$ and $z \longrightarrow z''$, must tend to the point $\tau_\nu z'' \in \partial D$ (since the mapping T_ν^{-1} is an automorphism of the domain D). The systems of functions defining the mappings τ_ν form a normal family in the domain D (for each index k; see formula (5.1)); these mappings converge uniformly in the domain V^* (to the identity mapping). Therefore the mappings τ_ν converge uniformly in the whole domain D. But then $\lim_{\nu\to\infty} \tau_\nu z'' = z''$ and the interior point of the domain turns out to be the limit of a sequence of its boundary points. Thus our supposition on the mapping T is incorrect. For the mapping T^{-1} we can carry out a similar argument.

Hence we must conclude that T is an automorphism of the domain D. The proof of Lemma 4 and therewith of Theorem 20.4 is finished.

We turn our attention to an important consequence of the Theorem 20.4 just proved, which will be useful in the following exposition:

THEOREM 20.5. *The set of automorphisms of a bounded domain leaving some point O fixed is compact.*

From the theory of normal families it follows that in this case one can choose a convergent sequence from every infinite subset of our mappings. Theorem 20.4 proved above implies that the limit mapping belongs to the set of automorphisms which leave the point O fixed.

We shall close the present subsection by formulating (without proof) another result of H. Cartan concerning the iteration of mappings.

THEOREM 20.6. *Let T be a generalized biholomorphic mapping of a bounded domain D into itself. If the sequence of mappings T^{p_ν} (here $T^p = \underbrace{T \circ \cdots \circ T}_{p \text{ times}}$, $p_1 < p_2 < \cdots$, $\nu = 1, 2, \cdots$) converges uniformly in the domain D to a nondegenerate mapping, then the mapping T is an automorphism of the domain D.*

This theorem is of special interest because it allows us to obtain without difficulty one of the possible generalizations of Schwarz's lemma.

THEOREM 20.7. *Let T be a generalized biholomorphic mapping of a bounded domain D into itself, leaving fixed a point $O \in D$ (the point O is not a branch point of the domain D). Then the modulus of the Jacobian of the mapping T at the point O cannot be larger than unity. It is equal to unity if and only if the mapping T is an automorphism of the domain D.*

PROOF. The first assertion of the theorem follows from the fact that the Jacobians of the mappings T^p $(p \longrightarrow \infty)$ are bounded under our conditions, since for the iteration of mappings their Jacobians are multiplied. The second assertion follows from Theorem 20.6: If the modulus of the Jacobian of the mapping T at the point O is equal to unity, every convergent sequence of the mappings T^{p_ν} $(p_\nu \longrightarrow \infty)$ has a nondegenerate mapping as its limit and accordingly T is an automorphism. Conversely, if T is an automorphism and leaves the point O fixed, then in view of Theorem 20.4 only an automorphism can be the limit of an arbitrary sequence T^{p_ν} $(p_\nu \longrightarrow \infty)$. This latter is possible only in the case when the modulus of the Jacobian of the mapping T at the point O is equal to unity.

4. Groups of automorphisms of a domain. Consider a bounded domain D over the space C^n having no interior branch points. H. Cartan [4] investigated the general properties of the set G of all automorphisms of such a domain.

First of all it is easy to see that the set G is a group, i.e., that the group axioms are satisfied for the totality of automorphisms of the domain.

H. Cartan established that the group G consists of: 1) The Lie group

$$(G^*) \qquad z_k = \phi_k(z_1, \cdots, z_n, a_1, \cdots, a_p), \; k = 1, \cdots n, \; (z_1, \cdots, z_n) \in D, \qquad (5.7)$$

where $a_1, \cdots, a_p$ are real essential parameters of the group. Their number $p \geq n(n+2)$; they take the values of the coordinates of points of the hyperball $\{a_1^2 + \cdots + a_p^2 < r^2\}$ in the space R_p of these parameters. The values $a_1 = \cdots \cdots = a_p = 0$ correspond to the identity transformation of the group. In relation (5.7) the ϕ_k are analytic functions for all of their variables. It is further to be noted that to the group G^* there belong those transformations of the group G which make up some sufficiently small neighborhood of the identity transformation.

2) A finite or countable series of subsets $T_1 \circ G^*$, $T_2 \circ G^*, \cdots$. Here $T_1, T_2, \cdots$ are automorphisms belonging to the group G, but not to the group G^*.

The case when G is a discrete group is exceptional.

The set $\mathfrak{G}$ of automorphisms of a bounded domain $D \subset C^n$ leaving some point $t \in D$ fixed plays an important role in the theory of biholomorphic mappings. This

set $\mathfrak{G}$ represents a subgroup of the group G (which can be verified immediately) and is called *the subgroup of stability or the stationary subgroup for the point* $t \in D$ *of the group of automorphisms of the domain* D.

We have proved above (see Theorem 20.5) that the group $\mathfrak{G}$ is compact.

In the case of one variable z, when D is a simply-connected domain distinct from the whole plane C_z^1, or of a plane with exceptional points, the group of automorphisms is a three-parameter Lie group; by the mapping of the domain D onto the unit disk it is reduced to the group of Möbius transformations. In the case of a finite doubly-connected domain D it is reduced to a one-parameter Lie group (if the domain D is conformally mapped onto a disk without its center), or to two continuous sets of mappings (if D is conformally mapped onto an annulus), one of which is a Lie group (rotations) and the other is obtained from the former by superposing its mappings on the mapping that inverts the given annulus. Multiply-connected domains may have only discrete groups of automorphisms.

In the sequel (see §23 and §24) we shall consider many classes of domains having groups of automorphisms.

§21. METRIC INVARIANT UNDER BIHOLOMORPHIC MAPPINGS OF DOMAINS OF THE SPACE C^n

We consider two methods of constructing metric invariants under biholomorphic mappings of domains of the space C^n. One of them is due to C. Carathéodory, and the other to S. Bergman.

1. The Carathédory metric. Let B be a bounded domain in the space C^n. We consider the set $\mathfrak{S}$ of functions $f(z) \in \mathfrak{O}_B$ which satisfy the condition $\sup_{z\in B} |f(z)| < 1$. In the domain B these functions evidently form a normal family. We denote by $E(z', z'')$ the Euclidean distance between two points z', z'' of the disk $|z| < 1$. It is well known that

$$E(z', z'') = \ln \frac{|z'' - z'| + |1 - z'\overline{z''}|}{\sqrt{(1 - z'\overline{z'})(1 - z''\overline{z''})}}. \tag{5.8}$$

We agree to call the quantity

$$D_B(\mathfrak{z}, \zeta) = \sup_{f\in\mathfrak{S}} E(f(\mathfrak{z}), f(\zeta)). \tag{5.9}$$

the *Carathéodory distance* between points $\mathfrak{z}, \zeta \in B$ (see Carathéodory [1]). It is evident that we always have $D_B(\mathfrak{z}, \zeta) = D_B(\zeta, \mathfrak{z})$ and $D_B(\zeta, \zeta) = 0$.

Carathéodory established the following properties of this function:

THEOREM 21.1. *The Carathéodory distance between any two points* $\mathfrak{z}$ *and* ζ *of the domain* B *is finite.*

PROOF. Evidently, it suffices to consider the distance

$$D_B(\mathfrak{z}, \zeta) = \lim_{n\to\infty} E(f_n(\mathfrak{z}), f_n(\zeta)) \tag{5.10}$$

for some sequence of functions $f_n(z)$, where $f_n \in \mathfrak{S}$. At an arbitrary point z of domain B we set:

$$\varphi_n(z) = \frac{f_n(z) - f_n(\mathfrak{z})}{f_n(z)\overline{f_n(\mathfrak{z})} - 1}.$$

The functions $\phi_n(z)$ are equal to zero at the point $\mathfrak{z}$ and also belong to the family $\mathfrak{S}$. Further, in view of the properties of the distance between points, we have

$$E(f_n(\mathfrak{z}), f_n(\zeta)) = E(0, \varphi_n(\zeta)) = \frac{1}{2}\ln\frac{1+|\varphi_n(\zeta)|}{1-|\varphi_n(\zeta)|}. \tag{5.11}$$

From the sequence of functions $|\phi_n(z)|$ we can select a subsequence which converges to a function $\phi(z)$, holomorphic in the domain B, such that $\max_{z\in B}|\phi(z)| \le 1$. Therefore $|\phi(\zeta)| \le 1$ and, since the point $\zeta(\zeta_1, \cdots, \zeta_n) \in B$, we may expand the function $\phi(z)$ in a power series of differences $z_k - \zeta_k$ in some neighborhood of ζ. If the function ϕ is constant, then $\phi(\mathfrak{z}) = \phi(\zeta) = 0$ and the theorem is proved. If the function ϕ is not constant in the domain B, then $|\phi(\zeta)| < 1$ by Theorem 3.11, (I). But in view of equalities (5.10) and (5.11) it is easy to see that

$$D_B(\mathfrak{z}, \zeta) = \frac{1}{2}\ln\frac{1+|\varphi(\zeta)|}{1-|\varphi(\zeta)|} < \infty.$$

Thus the theorem is proved.

THEOREM 21.2. *If the domain* B *is mapped biholomorphically onto a domain* B^*, *while the points* $\mathfrak{z}$, ζ *are carried into points* $\mathfrak{z}^*$, ζ^*, *then*

$$D_B(\mathfrak{z}, \zeta) = D_{B^*}(\mathfrak{z}^*, \zeta^*).$$

This conclusion follows immediately from the definition of $D_B(\mathfrak{z}, \zeta)$. Under the indicated mapping, the values in the domain B of each function f from the family $\mathfrak{S}$ coincide with those in the domain B^* of the corresponding function f^* from the family $\mathfrak{S}^*$ consisting of functions that are holomorphic and smaller than unity in modulus in the domain B^*.

THEOREM 21.3. *For every three points* z, $\mathfrak{z}$, $\zeta \in B$ *we have the inequality*

$$D_B(z, \mathfrak{z}) \leqslant D_B(z, \zeta) + D_B(\zeta, \mathfrak{z}). \tag{5.12}$$

PROOF. As we have seen, for example in the proof of Theorem 21.1, there always exists a function $f \in \mathfrak{S}$ such that

$$D_B(z, \mathfrak{z}) = E(f(z), f(\mathfrak{z})). \tag{5.13}$$

For non-Euclidean distances in the unit disk we have the triangle inequality

$$E(z', z'') \leqslant E(z', z''') + E(z''', z''). \tag{5.14}$$

On the other hand, in view of equality (5.9),

$$\begin{aligned} E(f(z), f(\zeta)) &\leqslant D_B(z, \zeta), \\ E(f(\zeta), f(\mathfrak{z})) &\leqslant D_B(\zeta, \mathfrak{z}). \end{aligned} \tag{5.15}$$

Putting $z' = f(z)$, $z'' = f(\zeta)$, $z''' = f(\mathfrak{z})$ in relation (5.14) and using relations (5.13) and (5.15), we obtain the inequality (5.12).

THEOREM 21.4.

$$\sup_{\mathfrak{z} \in B} D_B(\zeta, \mathfrak{z}) = \infty.$$

PROOF. By the conditions of the theorem, ζ is a fixed point, while $\mathfrak{z}$ is a moving point of the domain B. Let z be another fixed point of the domain B, and let $f \in \mathfrak{S}$ be a function such that $f(\zeta) = 0$;

$$D_B(\zeta, z) = \frac{1}{2} \ln \frac{1 + |f(\zeta)|}{1 - |f(\zeta)|}.$$

Then, if it were true, in contradiction to the assertion of the theorem, that we always have $D_B(\zeta, \mathfrak{z}) < M$ (M is some constant number), evidently we would have

$$\frac{1}{2} \ln \frac{1 + |f(\mathfrak{z})|}{1 - |f(\mathfrak{z})|} \leqslant D_B(\zeta, \mathfrak{z}) < M.$$

Hence it follows that

$$|f(\mathfrak{z})| < \frac{e^{2M} - 1}{e^{2M} + 1} = \theta < 1.$$

Then the function $f/\theta \in \mathfrak{S}$ and the following relation holds

$$E\left(\frac{1}{\theta} f(\zeta), \frac{1}{\theta} f(z)\right) = \frac{1}{2} \ln \frac{\theta + |f(z)|}{\theta - |f(z)|} > D_B(\zeta, z),$$

which is evidently impossible. The theorem is proved.

THEOREM 21.5 (a generalization of Schwarz's lemma for Pick's invariant form). *If B and G are two domains, $B \subset G$, and $\mathfrak{z}$, ζ are two points belonging to these domains, then*

$$D_G(\mathfrak{z}, \zeta) \leq D_B(\mathfrak{z}, \zeta).$$

This inequality follows immediately from the definition of the Carathéodory distance. Every function which is holomorphic and smaller than unity in modulus in the domain G will be holomorphic and smaller than unity in modulus in the domain B. But for the domain B other functions are contained in the class of functions $\mathfrak{E}$. Hence, in view of the definition of the Carathéodory distance, our assertion follows.

In closing the present subsection we shall enumerate (without proof) some more properties of the Carathéodory distance (see Carathéodory [1,2]).

The Carathéodory distance for the bicylinder $B = \{|w| < 1, |z| < 1\}$ between its two points $P(w_1, z_1)$ and $Q(w_2, z_2)$ is defined by the formula

$$D_B(P, Q) = \max[E(w_1, w_2), E(z_1, z_2)]. \tag{5.16}$$

For the hyperball $G = \{|w|^2 + |z|^2 < 1\}$ the Carathéodory distance between points $O(0, 0)$ and $P(w, z)$ is defined by the formula

$$D_G(O, P) = \frac{1}{2} \ln \frac{1 + \sqrt{w\bar{w} + z\bar{z}}}{1 - \sqrt{w\bar{w} + z\bar{z}}}. \tag{5.17}$$

In order to obtain from this formula the distance between two arbitrary points P, Q of the hyperball, it is necessary to apply to formula (5.17) the mapping (5.80) which transforms the hyperball into itself in such a way that the point O is carried into Q, while P remains fixed.[1)]

The indicatrix of the Carathéodory metric. We confine ourselves to the case of two variables. The *indicatrix of the Carathéodory metric* (like every other metric) for a point $\zeta_0 \in B$ is defined as the collection of points which are separated from ζ_0, on the rays t emanating from that point, by a distance equal to $\lim_{m\to\infty} r(\zeta_m, \zeta_0)/D_B(\zeta_m, \zeta_0)$ (here $\zeta_m \in t$, $\lim_{m\to\infty}\zeta_m = \zeta$ and $r(\zeta_m, \zeta_0)$ is the Euclidean distance between ζ_m and ζ_0). It turns out that such an indicatrix bounds a complete and convex disk domain with its center at the point ζ_0.

If ζ_0 is the center of some complete and convex disk domain B, then the

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1) By comparison of formulas (5.16) and (5.17) it can be concluded that the bicylinder and the hyperball cannot be mapped biholomorphically onto each other. This result was obtained in 1907 by H. Poincaré [1] and served as a starting point for the investigation of the theory of biholomorphic mappings.

indicatrix of the Carathéodory metric for this domain B at the point $\zeta_0 \in B$ coincides with the boundary of the domain B.

From what has been said it is clear that the Carathéodory metric is in general not Riemannian, since for the latter the indicatrix is always a hypersurface of the second order (a hyperellipsoid if the Riemann metric is positive definite). We further note that Theorem 21.5 of the present subsection may be expressed by means of the concept of an indicatrix as follows:

If $B \subset G$ and a point ζ_0 belongs to both domains, then the indicatrix of the point ζ_0 relative to the domain B lies inside the indicatrix of the point ζ_0 relative to the domain G.

The equivalence of the two above-stated propositions is established immediately.

In other papers the problem of the behavior of the function $D_B(\zeta, \mathfrak{z})$ when the point $\mathfrak{z}$ approaches the boundary of the domain (the point ζ being fixed) has been considered (Behnke-Thullen [2], pp. 101–102).

2. Properties of the Bergman invariant metric. The Kähler metric. In §5.1, Chapter I we defined, in a bounded domain $D \subset C^n$, the Bergman metric invariant under biholomorphic mappings: [1)]

$$ds^2 = T_{l\overline{m}}\, dz^l\, \overline{dz^m}, \tag{5.18}$$

where the components of the metric tensor $T_{l\overline{m}}$ are defined by the equations

$$T_{l\overline{m}} = \frac{\partial^2 \ln K_D(z, \bar{z})}{\partial z^l\, \overline{\partial z^m}}. \tag{5.19}$$

Here $K_D(z, \overline{z})$ is the kernel function of the domain D. It has been established that the form (5.18) is hermitian (namely $T_{l\overline{m}} = \overline{T_{m\overline{l}}}$) and positive definite.

We remark that the hermitian metric which is invariant under biholomorphic mappings can be defined in a more general way by the formula

$$ds_1^2 = I T_{l\overline{m}}\, dz^l\, \overline{dz^m}, \tag{5.20}$$

where I is an arbitrary positive invariant of the biholomorphic mappings.

The Bergman metric is a special case of the Kähler metric (see Kähler [1]).

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1) In the sequel we shall as usual omit the symbol Σ. For the summation unbarred indices take values $1, \cdots, n$ and barred indices values $\overline{1}, \cdots, \overline{n}$.

DEFINITION (Kähler metric). The positive definite hermitian metric given, in a domain $D \subset C^n$, by means of the formula

$$ds^2 = h_{l\overline{m}}\, dz^l \, \overline{dz^m}, \tag{5.21}$$

where the function $h_{l\overline{m}} \in \mathcal{C}^2$ in the domain D, is said to be *Kählerian* if there exists a function $\Phi \in \mathcal{C}^4$ in the domain D, such that

$$h_{l\overline{m}} = \frac{\partial^2 \Phi}{\partial z^l \, \overline{\partial z^m}}. \tag{5.22}$$

The function Φ in this case is called the *Kähler potential* of the hermitian metric (5.21). Because of the positive definiteness of the form (5.21), Φ is a plurisubharmonic function. Moreover we always have $\Phi \in \mathcal{C}^\omega$.

Instead of the Kähler metric we sometimes speak of the *Kähler geometry* defined by formulas (5.21) and (5.22).

Form (5.21) defines a *locally* Kählerian metric in the domain D if it possesses a Kähler potential in some neighborhood of each point $\zeta \in D$. It is easy to see that the metric (5.21) is locally Kählerian if and only if the exterior form

$$\Omega = i h_{l\overline{m}}\, dz^l \wedge \overline{dz^m}$$

is closed in the domain D.[1]

We now derive some formulas for a hermitian Kählerian geometry which will be needed in the sequel.

As we have already mentioned (see §8.4, Chapter I, (I)), every vector u^k ($k = 1, \cdots, n$; always $\overline{u^k} = u^{\overline{k}}$) uniquely defines a one-complex-dimensional plane U to which it belongs.

The angle ψ between vectors u^k and v^k in the hermitian metric (5.21) can be defined by the formula

$$\cos \psi = \frac{\operatorname{Re}(u^k, v^k)}{|u^k| \, |v^k|}, \tag{5.23}$$

where

$$(u^k, v^k) = h_{l\overline{m}} u^l v^{\overline{m}}, \tag{5.24}$$

$$|u^k| = (u^k, u^k)^{\frac{1}{2}}, \quad |v^k| = (v^k, v^k)^{\frac{1}{2}}.$$

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[1] See, for example, A. Weil, *Introduction à l'étude des variétés Kählériennes*, Hermann, Paris, 1958; Russian transl., IL, Moscow, 1961; Chapter II.

We take analytic planes U, V defined, respectively, by vectors u^k, v^k and consider, as in the Euclidean case, the minimum of the angles between vectors lying in these planes. This minimum value θ is called the angle between the planes U and V. Further, in the plane V we take a vector v'^k such that $\angle(u^k, v'^k) = \theta$. Then if we set $\angle(v^\alpha, v'^\alpha) = \phi$, we have (see Fuks [1], (I), [5], (I))

$$e^{i\varphi}\cos\theta = \frac{(u^k, v^k)}{|u^k|\,|v^k|}; \tag{5.25}$$

$$\sin\theta = \frac{\left(\sum\limits_{\substack{l,p,q,m=1\\ l<p,\,q<m}}^{n} \begin{vmatrix} h_{l\bar m} & h_{l\bar q} \\ h_{p\bar m} & h_{p\bar q} \end{vmatrix} \begin{vmatrix} u^l & u^p \\ v^l & v^p \end{vmatrix} \begin{vmatrix} u^{\bar m} & u^{\bar q} \\ v^{\bar m} & v^{\bar q} \end{vmatrix}\right)^{\frac{1}{2}}}{|u^k|\,|v^k|}. \tag{5.26}$$

Just as in the Euclidean case, the angles θ and ϕ are called the *first and second analytic* angles between the vectors u^k and v^k. The quantity $|u^k|\,|v^k|\sin\theta$ is called the *analytic* or *complex area* confined between the vectors u^k and v^k.

In the general case of m (> 2) vectors we may introduce the concept of the m-complex-volume enclosed by these vectors. It turns out by calculation that the following formula holds; it is a generalization of formula (5.26) (see Šmatkov [1]).

If we denote by Ψ, F, Φ the corresponding angles between the vectors u^k and v^k calculated in the Euclidean metric, then it has been shown that for $n = 2$ (see Fuks [1], (I))

$$\sin F = \frac{1}{\sqrt{D}}\,\frac{|u^k|\,|v^k|}{ab}\sin\theta,$$

$$\cos(\Phi - \varphi) = \frac{\left(\frac{|u^k|}{a}\right)^2 + \left(\frac{|v^k|}{b}\right)^2 - \left(h_{1\bar 1} + h_{2\bar 2}\right)\sin^2 F}{2\,\frac{|u^k|\,|v^k|}{ab}\cos\theta\cos F}. \tag{5.27}$$

Here $a^2 = |u^1|^2 + |u^2|^2$, $b^2 = |v^1|^2 + |v^2|^2$.

Now assume that the metric (5.21) under consideration is Kählerian. Then, by the use of relation (5.22), we find that the components of the curvature tensor of such a metric are calculated by the following formula:

$$R^p_{ml\bar q} = -\frac{\partial}{\partial \bar z^{\bar q}}\left\{ {p \atop ml} \right\}. \tag{5.28}$$

The remaining components of this tensor are either calculated as conjugates or are equal to zero. Here $\left\{ {p \atop ml} \right\}$ are Christoffel symbols characterizing the parallel

displacement in the geometry under study.

The Riemann curvature with respect to the topologically two-dimensional direction (plane) defined by the vector u^k is seen to be equal to

$$\frac{R_{\bar{q}lp\bar{m}}u^{\bar{q}}u^l u^p u^{\bar{m}}}{|u^k|^4} = R. \tag{5.29}$$

Here

$$R_{\bar{q}lp\bar{m}} = h_{r\bar{q}} R^r_{lp\bar{m}} = -\frac{\partial^2 h_{p\bar{q}}}{\partial z^l \partial \bar{z}^m} + h^{r\bar{s}} \frac{\partial h_{p\bar{s}}}{\partial z^l} \frac{\partial h_{r\bar{q}}}{\partial \bar{z}^m}. \tag{5.30}$$

Now assume that the metric under consideration is that of Bergman. Then $h_{l\bar{m}} = T_{l\bar{m}}$, where $T_{l\bar{m}}$ is defined by formula (5.19). By the use of equality (5.30), it can be shown that in this case

$$K_D |u^k|^4 (2 - R) = I^{-1}_{\min}, \tag{5.31}$$

where $K_D = K_D(z, \bar{z})$ is the kernel function of the domain D, and $I_{\min}$ is the minimum value of the integral $I = \|f\|^2_D$ (see §4.5, Chapter I) under the following supplementary conditions imposed on the function $f \in L^2(D)$ at some fixed point $z \in D$:

$$f(z) = \frac{\partial f}{\partial z^1} = \dots = \frac{\partial f}{\partial z^n} = 0,$$

$$\sum_{l,m=1}^{n} \frac{\partial^2 f}{\partial z^l \partial \bar{z}^m} u^l u^{\bar{m}} = 1$$

(the corresponding calculation in the case $n = 2$ has been carried out in §4.5, Chapter I: see formula (1.61)).

The quantity $I_{\min}$ is one of the values of the integral $I = \|f\|^2_D$ and therefore is always positive. Consequently, from formula (5.31) there follows the

THEOREM 21.5 (Fuks [1]). *The Riemann curvature of a Bergman geometry with respect to any two-dimensional analytic direction is always less than two.*

A generalization of this result was obtained by L. K. Hua [2].

3. The Kähler metric on a complex manifold represents a natural generalization of the Kähler metric in a domain of the space C^n. Let Z be a complex manifold with n as its complex dimension. On this manifold we define an hermitian Kähler tensor of the second order $h_{l\bar{m}}$. For this purpose we give on each chart (U, ψ) of the manifold Z the components of the tensor as functions of the local coordinates $z^1, \cdots, z^n$, subject to the condition $\overline{h}_{l\bar{m}} = h_{m\bar{l}}$. In that case, if (U^*, ψ^*) is another chart of the manifold Z, $U \cap U^* \neq \emptyset$, and $z^{*1}, \cdots, z^{*n}$ are

local coordinates on this chart, and $h^*_{l\overline{m}}$ are components of the given tensor on the chart (U^*, ψ^*), then $h^*_{l\overline{m}} = h_{p\overline{q}}\,(\partial z^p/\partial z^{*l})\,\overline{(\partial z^q/\partial z^{*m})}$.

By means of a tensor such as $h_{l\overline{m}}$ there is defined on the manifold Z the hermitian metric

$$ds^2 = h_{l\overline{m}}\, dz^l\, \overline{dz^m}; \tag{5.21$_1$}$$

here it is assumed that the form (5.21$_1$) is positive definite.

The hermitian metric (5.21$_1$) is said to be *Kählerian* if there exists on the manifold Z a function Φ (usually $\Phi \in \mathcal{C}^\omega$) such that for each point $z \in Z$ one can find a chart (U, ψ), $z \in U$, on which

$$h_{l\overline{m}} = \frac{\partial^2 (\Phi \circ \psi^{-1})}{\partial z^l\, \partial \overline{z}^m} \circ \psi. \tag{5.22$_1$}$$

Here $z^1, \cdots, z^n$ are local coordinates on the chart (U, ψ). The function Φ in this case is called the *Kähler potential* of the metric (5.21$_1$). Because of the positive definiteness of the form (5.21$_1$) the function Φ is plurisubharmonic.

The metric (5.21$_1$) is said to be *locally Kählerian* on the manifold Z if such a function as Φ is defined only on the indicated chart (U, ψ) (and, generally speaking, is not united into one function $\Phi \in \mathcal{C}^\omega$ on the whole manifold Z). It can be shown, just as in the case of a domain $D \subset C^n$, that the metric (5.21$_1$) is locally Kählerian if and only if the exterior form $\Omega = ih_{l\overline{m}}\, dz^l \wedge \overline{dz}^{\,m}$ is closed.

By the length of a line $L \subset Z$ in the metric (5.21$_1$) we understand the integral of the square root of the form (5.21$_1$) taken along that line. By the distance between points z, $\zeta \in Z$ we understand the greatest lower bound of the lengths of all lines $L \subset \mathcal{C}^1$ joining these points within the limits of the manifold Z. The metric (5.21$_1$) is said to be *complete* in the manifold Z if for every sequence of points $\zeta_\nu \in Z$, $\nu = 1, 2, \cdots$, having no points of accumulation on the manifold Z, one can find a point $\zeta_0 \in Z$ whose distance to the points of that sequence increases indefinitely as $\nu \longrightarrow \infty$. (It is easy to see that in fact the choice of the point $\zeta_0 \in Z$ has no influence on the definition of this property of the metric.) It can be shown that the metric (5.21) defined in a bounded domain $D \subset C^n$ is complete if and only if all the points of the boundary ∂D are infinitely distant in that metric.

The question of conditions for the completeness of the Kähler metric defined on the complex manifold Z (in particular, on the domain $D \subset C^n$) will be considered below in §25.5 of the present chapter.

4. **Curvature of analytic surfaces and hypersurfaces of the space C^2 in the Kähler metric.** We consider an analytic surface given in a neighborhood of an ordinary point $z \in D \subset C^n$ by the equation

$$z^2 = f(z^1),$$

where $f(z^1)$ is a holomorphic function in that neighborhood of the point z. Assume that the Kähler metric (5.21) is defined in the domain D.

At points $z(z^1, z^2)$ and $z'(z^1 + dz^1, z^2 + dz^2)$ of our surface we construct its tangent analytic planes and then make a parallel displacement (in the sense of the metric adopted) of the tangent plane from the point z' to the point z (either the vectors forming this plane are displaced or else the corresponding bivector). Then we find the angle $d\theta$ between this displaced analytic plane (the analytic property of the plane is preserved under the displacement) and the tangent analytic plane at the point z.

We call the quantity

$$\frac{d\theta}{ds} = \frac{\sqrt{D}\,Q}{S^{3/2}} \tag{5.32}$$

the *curvature* of the surface, just as in the Euclidean case (see §4.8, Chapter I, (I)). Here ds is the distance between the points z and z'; $D = h_{1\bar{1}}h_{2\bar{2}} - h_{1\bar{2}}h_{2\bar{1}}$,

$$S = h_{1\bar{1}} + h_{1\bar{2}}\bar{f}' + h_{2\bar{1}}f' + h_{2\bar{2}}f'\bar{f}';$$

$$Q = f'' + \left\{ {2 \atop 11} \right\} + f'\left(2\left\{ {2 \atop 12} \right\} - \left\{ {1 \atop 11} \right\}\right) + f'^2\left(\left\{ {2 \atop 22} \right\} - 2\left\{ {1 \atop 12} \right\}\right) - f'^3\left\{ {1 \atop 22} \right\}.$$

If a hypersurface of the class $\mathcal{C}^2$

$$\Phi(x^1, y^1, x^2, y^2) \equiv F(z^1, z^2) = 0$$

is given (see Mitrohin [1]), then its curvature may be characterized, just as in the Euclidean case (see §12.5, Chapter II, (I)), as follows: at each point z of such a hypersurface we take three mutually perpendicular directions which belong to the hyperplane tangent at z and to the hypersurface and form, together with the normal to the surface, the so-called normal system of coordinates. Recall that these directions are the following: two of them (T_1 and T_2) belong to the analytic plane passing through the point z and lying completely in the tangent hyperplane (such an analytic plane is uniquely determined); the third direction (N) is defined by the intersection of the tangent hyperplane with the analytic plane drawn through the

normal to the hypersurface.

We further take three points adjacent to the point z in the directions T_1, T_2, N, and then construct the normals at these points, displace them to the point z and form the six quantities:

$$\left(\frac{d\theta}{ds}\right)_{T_1}, \quad \left(\frac{d\theta}{ds}\right)_{T_2}, \quad \left(\frac{d\theta}{ds}\right)_{N}, \quad \left(\frac{d\varphi}{ds}\right)_{T_1}, \quad \left(\frac{d\varphi}{ds}\right)_{T_2}, \quad \left(\frac{d\varphi}{ds}\right)_{N} \tag{5.33}$$

(here $d\theta$, $d\phi$ are the above-mentioned analytic angles between the directions of the normals and ds is the distance from the point z to the corresponding adjacent point). These are also considered as quantities characterizing the curvature of the hypersurface (the components of the relative curvature tensor are expressed by their linear combinations).

The case when the hypersurface is analytic is interesting. By the use of the quantities (5.33), this situation may be expressed by the fact that in this case we have

$$\left(\frac{d\theta}{ds}\right)_{T_1} = \left(\frac{d\theta}{ds}\right)_{T_2} = 0, \quad \left(\frac{d\varphi}{ds}\right) = \frac{1}{3} H. \tag{5.34}$$

Here H is the so-called mean curvature of the hypersurface. In the general case

$$\left(\frac{d\varphi}{ds}\right)_{N} = \frac{1}{3} H + L(\Phi) \tag{5.35}$$

(here $L(\Phi)$ is Levi's determinant). As we see, equalities (5.34) and (5.35) do not differ from the corresponding equalities obtained above for the Euclidean metric.

5. Distances in the Bergman metric for the bicylinder and the hyperball. It is well known from differential geometry that the length of a geodesic line (we consider here the metric (5.18); $n = 2$) joining a fixed point, which we take as the origin of coordinates, to a certain moving point (ζ^1, ζ^2) of the domain D, namely the function $S(\zeta, 0)$ satisfies the equation

$$\nabla s = 4T^{m\bar{l}} p_m p_{\bar{l}} = 1. \tag{5.36}$$

Here ∇ is the first Beltrami operator and $p_m = \partial S/\partial \zeta^m$; we consider the function $S(\zeta, 0)$ in some sufficiently small neighborhood of the point $(0, 0)$ where it is uniquely defined.

In the case when D is the bicylinder $\{|z^1| < 1, |z^2| < 1\}$, by using formulas (1.55) and (5.19), we obtain, instead of (5.36), the equation

$$2(1 - \zeta^1\bar{\zeta}^1) p_1 p_{\bar{1}} + 2(1 - \zeta^2\bar{\zeta}^2) p_2 p_{\bar{2}} = 1. \tag{5.37}$$

By integrating this equation, [1] we are led to the following result:

$$S(\zeta, 0) = \sqrt{2}\sqrt{\ln^2\frac{1+|\zeta^1|}{1-|\zeta^1|} + \ln^2\frac{1+|\zeta^2|}{1-|\zeta^2|}}\,. \tag{5.38}$$

For the definition of the distance between two arbitrary points it suffices to make use of a mapping of the bicylinder onto itself, under which the origin of the coordinates is carried into an arbitrary point (z_0^1, z_0^2), while the point (ζ^1, ζ^2) is carried into another point (z^1, z^2). This mapping, evidently, may be taken in the form:

$$\zeta^k = \frac{z^k - z_0^k}{1 - z^k\overline{z_0^k}}, \qquad k = 1,\ 2. \tag{5.39}$$

Substituting ζ^k from (5.39) into formula (5.38), we obtain the *formula for the distance* $S(z, z_0)$, *in the Bergman metric, between two arbitrary points in the bicylinder.*

We again consider the hyperball $\{|z^1|^2 + |z^2|^2 < 1\}$ in the space C_z^2. In this case, by using formulas (1.54) and 5.19), we obtain, instead of (5.36), the equation

$$\frac{4}{3}(1-\zeta^1\overline{\zeta^1}-\zeta^2\overline{\zeta^2})\big[(1-\zeta^2\overline{\zeta^2})\,p_1p_{\bar 1}-\zeta^1\overline{\zeta^2}p_1p_{\bar 2} \\ -\zeta^2\overline{\zeta^1}p_2p_{\bar 1}+(1-\zeta^1\overline{\zeta^1})\,p_2p_{\bar 2}\big]=1. \tag{5.40}$$

Integrating this equation we arrive at the result:

$$S(\zeta, 0) = \frac{\sqrt{3}}{2}\ln\frac{1+\sqrt{\zeta^1\overline{\zeta^1}+\zeta^2\overline{\zeta^2}}}{1-\sqrt{\zeta^1\overline{\zeta^1}+\zeta^2\overline{\zeta^2}}}\,. \tag{5.41}$$

In order to calculate the distance between two arbitrary points, it is sufficient to make use of a mapping of the hyperball onto itself (see formulas (5.80)), under which the points (0, 0) and (ζ^1, ζ^2) are carried into the arbitrary points (z_0^1, z_0^2) and (z^1, z^2) of the hyperball. As a result we obtain:

$$S(z, z_0) \\ = \frac{\sqrt{3}}{2}\ln\frac{\left|1-z^1\overline{z_0^1}-z^2\overline{z_0^2}\right|+\sqrt{\left|z^1-z_0^1\right|^2+\left|z^2-z_0^2\right|^2-\left|z^1z_0^2-z^2z_0^1\right|^2}}{\left|1-z^1\overline{z_0^1}-z^2\overline{z_0^2}\right|-\sqrt{\left|z^1-z_0^1\right|^2+\left|z^2-z_0^2\right|^2-\left|z^1z_0^2-z^2z_0^1\right|^2}}\,. \tag{5.42}$$

By comparing formulas (5.38), (5.41) and (5.42) with the corresponding formulas for the Carathéodory geometry (see equalities (5.16) and (5.17)), we find that in the hyperball the two metrics coincide up to a factor; in the bicylinder they are

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[1] The intermediate calculations are to be found in the first edition of the present book.

already essentially different from each other. This result can be foreseen if one takes account of the fact that the indicatrix of the Carathéodory metric for points of the hyperball is a hypersurface of the second order, but this is not so for points of the bicylinder (even for its center).

From formulas (5.38) and (5.42) it is easy to conclude that the Bergman metric is complete in the bicylinder as well as in the hyperball.

§22. REPRESENTATIVE COORDINATES OF THE BERGMAN METRIC

1. **Completely geodesic analytic surfaces.** We consider first of all the condition under which an analytic surface will be completely geodesic in the Bergman metric (5.18) for a domain $D \subset C^n$. (A surface is said to be completely geodesic if all its geodesic lines are geodesics for the whole space.) For simplicity of writing we carry out all the calculations for the space C_z^2.

As is well known, in order that a certain surface should be completely geodesic, it is necessary and sufficient that all components of its relative curvature tensor $H_{ml}^{\cdot\cdot p}$ should be equal to zero. From the general formula for the tensor $H_{ml}^{\cdot\cdot p}$ it is easy to see that for an analytic surface

$$z^2 = f(z^1) \tag{5.43}$$

there follows the equality (see Fuks [1]):

$$H_{11}^{\cdot\cdot 1} = \frac{(T_{1\bar{1}} + \bar{f}' T_{1\bar{2}})\, Q}{S}; \quad H_{11}^{\cdot\cdot 2} = \frac{(T_{2\bar{1}} + \bar{f}' T_{2\bar{2}})\, Q}{S}; \tag{5.44}$$

$H_{\bar{1}\bar{1}}^{\cdot\cdot\bar{1}}$, $H_{\bar{2}\bar{2}}^{\cdot\cdot\bar{2}}$ are calculated as conjugates, and the remaining components of this tensor are equal to zero in our case. (Here the quantities S and Q are the same as in formula (5.32) but they are calculated in the metric (5.18).) Hence it is seen that a completely geodesic analytic surface is defined by the differential equation

$$Q = 0, \tag{5.45}$$

since both factors for Q in the expressions (5.44) cannot be simultaneously equal to zero, since formula (5.18) is positive definite and its discriminant $D = T_{1\bar{1}}T_{2\bar{2}} - T_{1\bar{2}}T_{2\bar{1}} \neq 0$.

Using equality (5.22) and the relations

$$\left\{ {1 \atop kl} \right\} = \frac{T_{2\bar{2}} \dfrac{\partial T_{l\bar{1}}}{\partial z^k} - T_{2\bar{1}} \dfrac{\partial T_{l\bar{2}}}{\partial z^k}}{D}, \quad \left\{ {2 \atop kl} \right\} = \frac{T_{1\bar{1}} \dfrac{\partial T_{l\bar{2}}}{\partial z^k} - T_{1\bar{2}} \dfrac{\partial T_{l\bar{1}}}{\partial z^k}}{D}$$

which are easily verified, and introducing the notations

$$\frac{d\psi}{dz^1}=\frac{\partial\psi}{\partial z^1}+f'\frac{\partial\psi}{\partial z^2},$$

$$\frac{d^2\psi}{(dz^1)^2}=\frac{\partial\psi}{\partial z^2}f''+\frac{\partial^2\psi}{(\partial z^1)^2}+2f'\frac{\partial^2\psi}{\partial z^1\,\partial z^2}+f'^2\frac{\partial^2\psi}{(\partial z^2)^2}$$

(where ψ is an arbitrary function of the class $\mathcal{C}^2$), we obtain after some calculations (omitted here) the equation

$$DQ=\frac{1}{K^3}\begin{vmatrix} K & K_{\bar z^1} & K_{\bar z^2} \\ \frac{dK}{dz^1} & \frac{dK_{\bar z^1}}{dz^1} & \frac{dK_{\bar z^2}}{dz^1} \\ \frac{d^2K}{(dz^1)^2} & \frac{d^2K_{\bar z^1}}{(dz^1)^2} & \frac{d^2K_{\bar z^2}}{(dz^1)^2} \end{vmatrix}=0. \tag{5.46}$$

Here $K_{\bar z 1} = \partial K/\partial \bar z^1$, $K_{\bar z 2} = \partial K/\partial \bar z^2$ and $K = K_D(z, \bar z)$ is the kernel function of the domain D. The left-hand side of the equation obtained has the form of the Wronskian for the functions K, $K_{\bar z 1}$, $K_{\bar z 2}$ and therefore the desired function $z^2 = f(z^1)$ must satisfy a relation of the form

$$\bar\alpha K+\bar\beta\frac{\partial K}{\partial\bar z^1}+\bar\gamma\frac{\partial K}{\partial\bar z^2}=0. \tag{5.47}$$

(Here $\bar\alpha$, $\bar\beta$, $\bar\gamma$ are appropriately chosen functions of the variables $\bar z^1$, $\bar z^2$ only.) Conversely, every analytic surface on which relation (5.47) holds is completely geodesic.

For some completely geodesic surface (5.43) with appropriate functions $\bar\alpha$, $\bar\beta$, $\bar\gamma$ of the variables $\bar z^1$, $\bar z^2$, let the condition (5.47) be identically satisfied. Since we must consider z^1 and $\bar z^1$ as independent variables in taking formal derivatives, we may replace, $\bar z^1$ by $\bar t^1$, without violating the identity, where the point t^1 is situated in some sufficiently small neighborhood of the point z^1. It is evident that in addition to equality (5.43) the conjugate equality $\bar z^2 = \overline{f(z^1)}$ is also satisfied in our surface. Therefore, in replacing $\bar z^1$ by $\bar t^1$, we must also replace $\bar z^2$ by $\bar t^2$; the point $t(t^1, t^2)$ will lie on our surface. Thus if the points $z(z^1, z^2)$ and $t(t^1, t^2)$ lie on the surface (5.43), then the following relations are fulfilled:

$$\overline{\alpha(t)}+\overline{\beta(t)}\frac{\partial\ln K(z,\bar t)}{\partial\bar t^1}+\overline{\gamma(t)}\frac{\partial\ln K(z,\bar t)}{\partial\bar t^2}=0,$$

$$\overline{\alpha(t)}+\overline{\beta(t)}\frac{\partial\ln K(t,\bar t)}{\partial\bar t^1}+\overline{\gamma(t)}\frac{\partial\ln K(t,\bar t)}{\partial\bar t^2}=0.$$

Subtracting one equation from another, we obtain

$$C_1 \frac{\partial \ln M_t}{\partial \bar{t}^1} + C_2 \frac{\partial \ln M_t}{\partial \bar{t}^2} = 0. \tag{5.48}$$

Here $M_t = K(z, \bar{t})/K(t, \bar{t})$ is the minimum function of the domain D for the point t (see (1.50)); C_1 and C_2 are constants. Note that the point t of each analytic plane is touched by one of the surfaces (5.48). This follows from the fact that $D \neq 0$. Thus we arrive at the following result:

THEOREM 22.1 (Fuks [1]). *Every completely geodesic analytic surface passing through the point $t(t^1, t^2)$ belongs to the family of surfaces* (5.48). *If in the direction of some analytic plane there passes through the point t a completely geodesic analytic surface, then it is the surface* (5.48) *that is tangential to this analytic plane at the point t.*

Theorem 22.1 also remains valid for the space C^n with $n > 2$; however, in this case equation (5.48) is replaced by the equations

$$\frac{\partial \ln M_t}{\partial \bar{t}^1} + C_k \frac{\partial \ln M_t}{\partial \bar{t}^k} = 0, \quad k = 2, \ldots, n.$$

We note that at the point t the surfaces (5.48) are always geodesic, i.e., all the components of the tensor $H^{\cdot\cdot p}_{ml}$ for such surfaces are equal to zero at the point t.

2. Representative coordinates. Mapping onto a representative domain. It is of great importance to consider the coordinate system in which the coordinates of the surface passing through a point t would belong to the surfaces (5.48), while the latter would be defined by linear equations. If it is also required that the functions which realize the passage to these coordinates

$$u^k = u^k(z^1, z^2), \quad k = 1, 2, \tag{5.49}$$

should be normalized with respect to the point $t(t^1, t^2)$, i.e., that the conditions

$$u^k(t^1, t^2) = 0, \quad \left(\frac{\partial u^k}{\partial z^l}\right)_t = \begin{cases} 0 & (k \neq l), \\ 1 & (k = l), \end{cases} \tag{5.50}$$

should be satisfied, then, by carrying out the corresponding calculations, we obtain for the functions (5.49):

$$u^k = T^{k\bar{l}}_{(t)} \frac{\partial \ln M_t}{\partial \bar{t}^l}. \tag{5.51}$$

DEFINITION (representative coordinates and domains). The representative coordinate system in a bounded domain $D \subset C^n$ with its center at a point $t \in D$ is defined by means of equalities (5.51). There $M_t = K(z, \bar{t})/K(t, \bar{t})$ is the minimum

function of the domain D and $T^{k\bar{l}}$ are the contravariant components of the fundamental tensor of the Bergman metric calculated at the point t (see equality (5.72) below for another expression of the functions u^k which is important in some applications).

Instead of introducing in the domain D the representative coordinates with their center at the point t, we sometimes consider the domain (generally speaking, over the space C^n) onto which the functions (5.51) map the domain D. This domain is said to be *representative* for the domain D relative to the point t. The mapping onto a representative domain was considered first by S. Bergman on the basis of another argument.[1]

Representative coordinates play the role of an analytic substitute for normal Riemannian coordinates at the point t. If we make use of them, it turns out that at the point t (for the space C^2) we have

$$\left\{ {1 \atop 22} \right\} = 2\left\{ {1 \atop 12} \right\} - \left\{ {2 \atop 22} \right\} = 2\left\{ {2 \atop 12} \right\} - \left\{ {1 \atop 11} \right\} = \left\{ {2 \atop 11} \right\} = 0. \tag{5.52}$$

(It is evident that the introduction of ordinary normal coordinates is realized by means of a nonanalytic transformation.)

We remark that in the case of one complex variable the transformations, analogous to (5.51), for simply-connected domains are reduced to the function $w = f(z)$ conformally mapping the given domain onto the disk whose radius is determined by the normalization[2] conditions $f(t) = 0$, $f'(t) = 1$.

In the case of the introduction of ordinary normal coordinates all the first derivatives of the fundamental tensor vanish at the origin. In our case we have the following situation: the equalities (5.52) can be rewritten in invariant form as

$$\left\{ {\alpha \atop \beta\gamma} \right\} + \frac{1}{3}\left(\delta_\beta^\alpha \Gamma_\gamma + \delta_\gamma^\alpha \Gamma_\beta\right) = 0. \tag{5.53}$$

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1) See Bergman [3], p. 432. In this paper the formulas for the functions u^k are derived in a different form: the point t is taken as the origin of coordinates. However, it is easy to establish that they are identical with formulas (5.51).

2) Note that the function mapping a simply-connected domain onto a disk may also be defined as $\int_t^z M(z, t)dz$ (see, for example, I. I. Privalov, *Introduction to the theory of functions of a complex variable*, Chapter XIII, §4 (Russian)). In such a case this function coincides with $T^{1\bar{1}}_{(t)} \partial \ln M_t / \partial \bar{t}$, a fact which is explained by the constant curvature of the invariant metric in this simplest case. In the general case such a coincidence does not occur.

Here δ^{α}_{β} is equal to zero for $\alpha \neq \beta$ and unity for $\alpha = \beta$; $\Gamma_{\alpha} = \partial \ln D/\partial z^{\alpha}$; these equalities hold for any dimension n.

Hence, by making use of the fact that

$$\frac{\partial T_{\alpha\bar{\beta}}}{\partial z^{\gamma}} = T_{\mu\bar{\beta}}\left\{ {\mu \atop \alpha\gamma} \right\},$$

we have

$$\frac{\partial T_{\alpha\bar{\beta}}}{\partial z^{\gamma}} = -\frac{1}{3}\left(T_{\alpha\bar{\beta}}\Gamma_{\gamma} + T_{\gamma\bar{\beta}}\Gamma_{\alpha}\right).$$

Thus in this case all the first derivatives are expressed by the quantities Γ_k. An analogous relation is also valid for the conjugate quantities.

Further consideration of the properties of the surfaces (5.48) allows us to establish that the case when one-complex-dimensional completely geodesic analytic surfaces actually exist is exceptional. We agree to say that there exists in the domain D a *complete system* of one-complex-dimensional completely geodesic analytic surfaces if such a surface passes through each point of the domain D in the direction of each one-complex-dimensional analytic plane. Then it turns out that we have the

THEOREM 22.2 (Fuks [1]). *In order that there should exist in the domain D a complete system of one-complex-dimensional completely geodesic analytic surfaces, it is necessary and sufficient that the geometry defined by formula* (5.18) *should be the geometry with constant Riemann curvature in the directions of all one-complex-dimensional analytic planes. Such a geometry is defined by the condition*

$$R_{\bar{\nu}\lambda\mu\bar{\omega}} = k\left(T_{\lambda\bar{\nu}}T_{\mu\bar{\omega}} + T_{\lambda\bar{\omega}}T_{\mu\bar{\nu}}\right), \qquad k = \text{const.} \tag{5.54}$$

It was first investigated by Schouten and van Dantzig, starting from the projective metrization of Study and Fubini (see Schouten and van Dantzig [1,2]), but more simply it can be defined by requiring the coincidence of the Riemann curvature (5.29) in the directions of all one-complex-dimensional analytic planes at each point (hence there already follows the coincidence of its values at all points of the space).

The condition (5.54) is satisfied and a similar geometry is realized for domains $D \subset C^n$ which are mapped biholomorphically onto the hyperball $\{|z^1|^2 + \cdots + |z^n|^2 < 1\}$. In the case of those domains of the space C^n which are mapped biholomorphically onto the bicylinder $\{|z^1| < 1, |z^2| < 1\}$, there passes

through each point of such a domain a family of one-complex-dimensional completely geodesic analytic surfaces depending on one real (not complex) parameter.

If a domain $D \subset C^n$ is itself representative, then by formula (5.51),

$$z^k \equiv T^{kl}_{(t)} \frac{\partial \ln M_t}{\partial \overline{t}^l} \tag{5.55}$$

(where the point t now coincides with the origin of coordinates) or

$$\frac{\partial \ln M_t}{\partial \overline{t}^l} = T^{(t)}_{r\overline{l}} z^r.$$

Differentiating both sides of the last equality with respect to z^k, we find that in our case

$$T^{(D)}_{k\overline{l}}(z, \overline{t}) = T^{(D)}_{k\overline{l}}(t, \overline{t}), \tag{5.56}$$

where $T^{(D)}_{k\overline{l}}(z, \overline{t}) = \partial^2 \ln K_D(z, \overline{t}) / \partial z^k \partial \overline{t}^l$. Conversely, equality (5.55) is easily obtained from equality (5.56). Thus we obtain the

THEOREM 22.3. *The conditions* (5.56) *are necessary and sufficient in order that the domain $D \subset C^n$ should be representative with respect to the point $t \in D$.*

Evidently our results concerning representative coordinates hold for the Kähler metric with a holomorphic potential $\Phi(z, \overline{t})$ (this last property ensures the correctness of the proof of formula (5.48)). The following theorem is concerned only with the Bergman geometry.

From formulas (5.51) and Theorem 5.1 there immediately follows the

THEOREM 22.4 (Bergman [3]). *If there exists for bounded domains $D_1, D_2 \subset C^n$ a biholomorphic mapping Π: $D_1 \longrightarrow D_2$, while $t_2 = \Pi t_1$, where the point $t_1 \in D_1$ and the point $t_2 \in D_2$, then the representative domains of the domains D_1 and D_2 relative to the points t_1 and t_2 coincide.*

3. Group of motions of the Bergman metric. We shall seek infinitesimal transformations (see Fuks [2])

$$Xf = \xi^k \frac{\partial f}{\partial z^k} + \overline{\xi}^k \frac{\partial f}{\partial \overline{z}^k},$$

$$\delta z^k = \xi^k(z^1, z^2)\,\delta p, \quad \delta \overline{z}^k = \overline{\xi}^k\,\delta p \tag{5.57}$$

(where p is a parameter) satisfying the condition

$$\delta(ds^2) = 0, \tag{5.58}$$

and the Lie group defined by them in a neighborhood of some point t of a given bounded domain $D \subset C^n$. Here ds^2 is defined by formula (5.18) (the case which will be important in the sequel); our results, however, also remain valid for the metric (5.21) with the condition that $\Phi \in \mathcal{C}^\omega$.

Such a group is called the *group of motions* for the corresponding geometry. Starting from formula (5.57), and keeping in mind the fact that $\delta(dz^k) = d(\delta z^k)$, we transform relations (5.58) and, by setting equal to zero the coefficient of $dz^i dz^k \delta p$ in the expression thus obtained, we arrive at the equality

$$\frac{\partial T_{i\bar{k}}}{\partial z^s}\xi^s + \frac{\partial T_{i\bar{k}}}{\partial \bar{z}^s}\bar{\xi}^s + T_{s\bar{k}}\frac{\partial \xi^s}{\partial z^i} + T_{i\bar{s}}\frac{\partial \bar{\xi}^s}{\partial \bar{z}^k} = 0.$$

Hence, by the use of formulas (5.19), we obtain

$$\frac{\partial \xi_i}{\partial \bar{z}^k} + \frac{\partial \xi_{\bar{k}}}{\partial z^i} = 0; \qquad \xi_i = T_{i\bar{s}}\bar{\xi}^s. \tag{5.59}$$

Killing's equations take such a form in this case. On the basis of equalities (5.19) we have

$$\xi_k = \frac{\partial \psi}{\partial z^k}; \qquad \xi_{\bar{k}} = \frac{\partial \bar{\psi}}{\partial \bar{z}^k},$$

where

$$\psi = \bar{\xi}^s \frac{\partial \ln K}{\partial \bar{z}^s}.$$

Now it is clear that equations (5.59) may be written in the form

$$\frac{\partial^2 (\psi + \bar{\psi})}{\partial z^i \partial \bar{z}^k} = 0, \qquad i, k = 1, \ldots, n.$$

We have thus arrived at the following theorem:

THEOREM 22.5. *In order that the infinitesimal transformation Xf should determine the group of motions of the Bergman geometry in a bounded domain $D \subset C^n$, it is necessary and sufficient that the function $X(\ln K_D)$ should be plurisubharmonic.*

Starting from this, we set up the problem of determining the quantities $\xi^k(z)$. By Theorem 22.5 one can write:

$$\xi^s(z)\frac{\partial \ln K(z, \bar{z})}{\partial z^s} + \overline{\xi^s(z)}\frac{\partial \ln K(z, \bar{z})}{\partial \bar{z}^s} = \varphi(z) + \overline{\varphi(z)}, \tag{5.60}$$

where $\phi(z)$ is a holomorphic function in some neighborhood of the point t. In identity (5.60) all variables z^k, $\bar{z}^k$ should be considered as independent. Therefore the identity is not violated when we put $z^k = t^k$ or $\bar{z}^k = \bar{t}^k$. If we subtract from the sum of the two identities obtained by these substitutions the identity obtained from equality (5.60) by the simultaneous substitution of z^k, $\bar{z}^k$ by t^k, $\bar{t}^k$, then we again obtain an identity. In this last identity we again put $\bar{z}^k = \bar{v}^k$, where $v(v^1, \cdots, v^n)$ is another point of the neighborhood of the point t under consideration. Finally we are led to the relation

$$\xi^s(z)\frac{\partial \ln \frac{K(z,\bar{t})}{K(z,\bar{v})}}{\partial z^s} + \xi^s(t)\frac{\partial \ln \frac{K(t,\bar{v})}{K(t,\bar{t})}}{\partial t^s} + \overline{\xi^s(v)}\frac{\partial \ln \frac{K(t,\bar{v})}{K(z,\bar{v})}}{\partial \bar{v}^s} + \overline{\xi^s(t)}\frac{\partial \ln \frac{K(z,\bar{t})}{K(t,\bar{t})}}{\partial \bar{t}^s} = 0.$$

We now put $\bar{v}^k = \bar{t}^k$ for $k \neq l$ and $\bar{v}^l = \bar{t}^l + d\bar{t}^l$, obtaining

$$-\xi^s(z)\frac{\partial^2 \ln K(z,\bar{t})}{\partial z^s\,\partial \bar{t}^l} + \xi^s(t)\frac{\partial^2 \ln K(t,\bar{t})}{\partial t^s\,\partial \bar{t}^l} - \frac{\partial}{\partial \bar{t}^l}\left(\overline{\xi^s(t)}\frac{\partial \ln M_t}{\partial \bar{t}^s}\right) = 0. \tag{5.61}$$

Contracting this expression with $T^{p\bar{l}}(z, \bar{t}) = \tau^{p\bar{l}}$, we obtain the

THEOREM 22.6. *If the infinitesimal transformation* $Xf = \xi^p(\partial f/\partial z^p) + \bar{\xi}^p(\partial f/\partial \bar{z}^p)$ *determines the group of motions of the Bergman geometry, then*

$$\xi^p(z) = \alpha^s\tau^{p\bar{l}}T^{(t)}_{s\bar{l}} - \bar{\alpha}^s\tau^{p\bar{l}}\frac{\partial^2 \ln M_t}{\partial \bar{t}^s\,\partial \bar{t}^l} + \beta^{\bar{s}}_{\bar{l}}\tau^{p\bar{l}}\frac{\partial \ln M_t}{\partial \bar{t}^s},$$

where α^s *and* $\beta^{\bar{s}}_{\bar{l}}$ *are appropriately chosen constants.*

We now consider the subgroup of the group of motions which leaves fixed a point $t \in D$, the so-called *subgroup of stability* or *stationary subgroup for the point t of the group of motions of the Bergman geometry.* Then, on account of the fact that $\xi^s(t) = 0$, we obtain from equality (5.61)

$$\xi^p(z) = \beta^{\bar{s}}_{\bar{l}}\tau^{p\bar{l}}\frac{\partial \ln M_t}{\partial \bar{t}^s}.$$

Hence from equality (5.57) we have

$$\delta\frac{\partial \ln M_t}{\partial \bar{t}^r} = \beta^{\bar{s}}_{\bar{l}}\frac{\partial \ln M_t}{\partial \bar{t}^s}\delta p.$$

Contracting both sides with $T_t^{s\bar{l}}$, we get by virtue of (5.51)

$$\delta u^s = \gamma_q^s u^q \delta p, \quad \frac{du^s}{dp} = \gamma_q^s u^q.$$

Here $\gamma_q^s = T_t^{s\bar{r}} \beta_{\bar{r}}^{\bar{m}} T_{q\bar{m}}^{(t)}$.

The solution of such a system of linear equations with constant coefficients has the form

$$u^r = A_s^r u_0^s,$$

where $u^s\big|_{p=0} = u_0^s$. Thus we obtain (by performing the substitution inverse to (5.51)) the following

THEOREM 22.7. *Transformations of the subgroup of stability for the point $t \in D$ of the group of motions of the Bergman geometry in that domain have the form*

$$\frac{\partial \ln M_t(z)}{\partial \bar{t}^r} = B_{\bar{r}}^{\bar{s}} \frac{\partial \ln M_t(z_0)}{\partial \bar{t}^s}.$$

When representative coordinates are introduced, these transformations turn out to be affine.

The last result again leads us to the representative coordinates. In essence, it is a generalization of one of H. Cartan's results [2], (I), which was investigated by Aronszajn [1]. Namely, it is clear that the group of motions of the Bergman geometry in a bounded domain D contains biholomorphic mappings of the domain D onto itself. This result of H. Cartan exactly represents Theorem 22.7 for that case (see Theorem 22.8 and §23).

Furthermore, starting from Theorems 22.5 and 22.6, we can obtain criteria for the existence of a p-nomial group of motions and its various subgroups.

Another application of the theory consists in the effective enumeration of all possible types of groups of motions, and in determining the corresponding expressions of the kernel functions of the domain D (see Fuks [2]).

Now we consider an application of the representative coordinates to locally isometric mappings of one domain of the space C^n onto another (see Fuks [3]). Consider, besides a bounded domain $D \subset C_z^n$, another bounded domain $D^* \subset C_w^n$; let $K^*(w, \bar{w})$ be the kernel function of the domain D^* and let the form

$$ds^2 = S_{m\bar{k}}\, dw^m\, d\bar{w}^k, \quad S_{m\bar{k}} = \frac{\partial \ln K^*}{\partial w^m\, \partial \bar{w}^k}$$

define the Bergman metric in the domain D^*.

We agree to call a biholomorphic mapping of a certain neighborhood of a point $a \in D$ onto a neighborhood of a point $b \in D^*$ a *locally isometric mapping of the bounded domain* D at the point a onto the bounded domain D^* at the point b if under this mapping $a \longrightarrow b$ and in the above-mentioned neighborhoods we have the equality

$$S_{m\bar{k}} = T_{q\bar{r}} \frac{\partial z^q}{\partial w^m} \frac{\partial \bar{z}^r}{\partial \bar{w}^k}. \tag{5.62}$$

The mapping of the whole domain D onto the domain D^* is a special case of such mappings. Let u^k, v^k be representative coordinates in the domains D, D^* relative to the points a and b, respectively. Then, putting $\bar{z}^k = \bar{a}^k$, $\bar{w}^k = \bar{b}^k$ in equations (5.62), we can integrate them and arrive at the following

THEOREM 22.8 (Fuks [3]). *Every locally isometric mapping of a domain* $D \subset C^n$ *at a point* $a \in D$ *onto a domain* $D^* \subset C^n$ *at a point* $b \in D^*$ *can be represented in the form*

$$u^k = \alpha_s^k v^s, \tag{5.63}$$

where the α_s^k *are appropriately chosen constants.*

This theorem contains Theorem 22.7 as a special case. It follows from it, in particular, that if for a locally isometric mapping of the domain D onto itself the point $a \in D$ is carried into the point $b \in D$, then this mapping is represented in representative coordinates (with centers at these points) in the form (5.63). Finally, everything is also valid in the special case of a biholomorphic mapping of the domain D onto the domain D^*, or for a biholomorphic mapping of the domain D onto itself, i.e., for an automorphism of the domain D. In particular, if the point a remains fixed under such an automorphism, then we must put $u^k = v^k$ in equality (5.63) and obtain the

COROLLARY OF THEOREM 22.8. *If an automorphism* T *of a bounded domain* $D \subset C^n$ *leaves fixed a point* $t \in D$, *then, by introducing in the domain* D *the representative coordinates* u^k *with their center at the point* t, *the automorphism* T *is reduced to the affine mapping*

$$u^{*k} = \alpha_s^k u^s, \tag{5.64}$$

where u^k, u^{*k} *are representative coordinates of some point of the domain* D *before and after the mapping.*

REMARK. In Theorem 22.7 the question is the subgroup of stability or the stationary subgroup for the point $t \in D$ of the group of motions, which is the Lie group generated by the infinitesimal transformations (5.57). In the present corollary

no assumption is made that the automorphism T belongs to some particular family of mappings.

§23. REPRESENTATIVE DOMAINS OF THE SPACE C^2

1. General properties. We shall consider the general properties of representative domains in the space C^n, where n is an arbitrary natural number. We carry out the classification of these representative domains only for the case $n = 2$.

As has been already noted, the mapping (5.51) of some bounded domain $D \subset C^n$ onto a representative domain is, in general, not biholomorphic. Different points of the domain D may have identical representative coordinates; the functions in the right member of equality (5.51) are not holomorphic, but only meromorphic in the domain D. Therefore a representative domain must be represented as a domain over some extended space.

Moreover, it is evident that the point corresponding to the origin of the coordinates cannot be an interior branch point. This follows from the fact that by virtue of the conditions (5.50) the Jacobian $\partial w/\partial z\big|_{z^k=0} = 1$. Hence it may also be concluded that the mapping onto a representative domain cannot be degenerate. However, the presence of exceptional manifolds is, in general, not excluded.

In what follows we shall denote a representative domain for the domain D relative to a point $t \in D$ by the symbol $R(D, t)$.

Our problem is to find fundamental types of representative domains which have families of automorphisms leaving fixed a certain point of the domain. To this end we shall first prove one auxiliary theorem:

THEOREM 23.1 (H. Cartan [2], (I)). *If the origin of coordinates belongs to a bounded domain $D \subset C^n$, then every one of its automorphisms T: $z \longrightarrow w$, normalized by the conditions* (5.50), *is the identity mapping.*

PROOF. Evidently in this case, instead of equalities (5.64), we obtain

$$u^k(z) = u^k(w), \tag{5.65}$$

where the point $z \in D$ is carried into the point $w \in D$ under the transformations in the theorem; instead of u^{*k}, u^k in formulas (5.64) we have written $u^k(z)$, $u^k(w)$.

We consider neighborhoods V, V_1 ($V_1 \subset V$) of the origin of coordinates which are so chosen that: 1) the Jacobian $\partial w/\partial z$ does not vanish in the domain V (this is always possible since the Jacobian is equal to unity at the origin of coordinates); 2) the fact that the point $z \in V_1$ implies that the point $w \in V$ (this is possible

since the origin remains fixed under the transformations involved). Then the mapping (5.51) is univalent in the domain V and it follows from (5.65) that the points z and w coincide with each other.

The holomorphic function $w^k(z)$ in the domain D coincides with z^k in the domain $V_1 \subset D$ $(k = 1, \cdots, n)$. Consequently, in general

$$w^k(z) \equiv z^k$$

in the domain D. Hence our theorem is proved.

Let G be the subgroup of stability for a point $t \in D$ of the group of automorphisms of a domain D, where D is a bounded domain of the space C^n. We take the point t as the origin of coordinates and set $z^* = gz$, if the transformation $g \in G$ and the point $z \in D$. To the group G in the domain $R(D, t)$ there corresponds a certain collection Γ of transformations of the form (5.64):

$$(\Sigma) \qquad u^{*k} = a_s^k u^s. \tag{5.66}$$

We shall show that these transformations have the following properties:

1) $$|\mathrm{Det}\, a_s^k| = 1.$$

In fact, it is evident that

$$|\mathrm{Det}\, a_s^k| = \left|\frac{\partial u^*}{\partial z^*}\right| \cdot \left|\frac{\partial z^*}{\partial z}\right| \cdot \left|\frac{\partial z}{\partial u}\right|. \tag{5.67}$$

Since $\mathrm{Det}\, a_s^k = \mathrm{const}$, it suffices to evaluate the right member of equality (5.67), for example, at the point t. At this point the last determinant of the right member is equal to unity by reason of the conditions (5.50). The middle determinant is equal to unity in view of equality (1.66) together with the fact that the transformations of the domain D do not affect the value of the kernel function of this domain at the point t.

2) *The form*

$$T_{k\bar{l}}^{(t)} u^k \bar{u}^l \tag{5.68}$$

is invariant under the transformations from Γ.

In fact, the form (5.18) is invariant under transformations $g \in G$. At the point t we have $dz^k = du^k$. In the transformations from Γ, one may substitute u^k for du^k, since these transformations are linear.

3) *The transformations from* Γ *together form a compact group.*

First of all we shall show that the transformations from Γ form a group isomorphic to the group G. In fact, by the conditions (5.50), the transformation $S \in G$

corresponding to the transformation $\Sigma \in \Gamma$ has the form

$$z^{*k} = a_s^k z^s + \cdots \tag{5.69}$$

(where the dots indicate terms of higher orders).

The transformation Σ cannot correspond to the two transformations $S, S' \in G$ since then the linear terms of these transformations in the expressions of the form (5.68) would coincide and the transformation $S' \circ S^{-1}$ would be the identity transformation, as follows from Theorem 23.1. But this is impossible since S and S' have been assumed to be different transformations. Further, from what has been said, it is clear that if transformations Σ_1 and Σ_2 from the collection Γ correspond to transformations $S_1, S_2 \in G$, then the transformation $\Sigma_1 \circ \Sigma_2$ corresponds to the transformation $S_1 \circ S_2$. Thus the collection Γ really represents a group isomorphic to the group G.

Since G is compact in view of Theorem 20.5, the group Γ is also compact.

2. Classification. In what follows we confine our attention to domains of the space C^2. Let us start from the fact that the group Γ consists of the transformations (5.66) with the above-mentioned three properties. We may always assume that by means of some affine transformation (whose determinant is equal to unity in modulus) the form (5.68) is reduced to the canonical form $|u^1|^2 + |u^2|^2$. Supposing that the coordinates z^k in the domain D have already been chosen correspondingly, we consider the new group γ obtained from the group Γ after replacing each of the transformations (5.66) by the two transformations:

$$(\gamma) \qquad v^k = \alpha^k u^1 + \beta^k u^2, \quad v^k = -(\alpha^k u^1 + \beta^k u^2),$$

where $\alpha^k = a_1^k e^{i\theta}$, $\beta^k = a_2^k e^{i\theta}$; the number θ being so chosen that $\alpha^1\beta^2 - \alpha^2\beta^1 = 1$. The group γ is evidently again compact and leaves the form $|u^1|^2 + |u^2|^2$ invariant.

The above conditions which are satisfied by the parameters $\alpha^1, \beta^1, \alpha^2, \beta^2$, can be put in geometric form. They indicate that the transformation of the τ-plane

$$\tau^* = \frac{\alpha^1\tau + \beta^1}{\alpha^2\tau + \beta^2}$$

is reduced, when the plane is stereographically projected onto a sphere, to a rotation of the sphere about one of its diameters.[1] Thus the group γ is isometric to

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[1] Concerning the necessary conditions for this result, see, for example, L. R. Ford, *Automorphic functions*, ONTI, 1938, pp. 131–133 (Russian transl.).

some compact group H of rotations of a sphere. As is well known, the following three cases are possible:

1) the group H consists of a finite set of transformations;

2) the group H is a group of rotations about one diameter (possibly supplemented by one more rotation through 180° about any diameter perpendicular to the basic diameter; such a rotation, evidently, can always be included in the above-mentioned group);

3) the group H unites all rotations of the sphere.

By analyzing the possible types of the group γ corresponding to the above possibilities of the group H (see H. Cartan [2], (I), pp. 65–68) and then by considering the possible types of the group Γ corresponding to the group γ thus obtained, one is led to the following final result:

THEOREM 23.2 (H. Cartan [2], (I)). *To a group G of automorphisms of a bounded domain $D \subset C^2$ which leaves fixed some point t of this domain, there corresponds in its representative domain $R(D, t)$ an isomorphic group Γ of homogeneous affine automorphisms. If the coordinate system z^1, z^2 is chosen properly in the domain D (the expression for $T^{(p)}_{n\overline{m}} q^n q^{\overline{m}}$ takes the form $|q^1|^2 + |q^2|^2$ in that system), then the following cases are possible:*

1) *The group Γ consists of a finite set of mappings.*[1)]

2) *The group Γ consists of the mappings*

$$u^{*1} = u^1 e^{im\theta};\ u^{*2} = u^2 e^{ip\theta} \tag{5.70}$$

and accordingly depends on one parameter. Here m and p are integers with no common factors. Aside from the mappings (5.70), there may still belong to the group Γ a finite set of homogeneous affine mappings with Jacobians equal to unity in modulus. The latter, for $|mp| \neq 1$, have the form

$$u^{*k} = u^k e^{2is\pi/n} \qquad (s = 1, \cdots, n;\ n \text{ being fixed}).$$

If $mp = -1$, then only one mapping

$$u^{*1} = u^1 e^{i\pi/n};\ u^{*2} = u^2 e^{i\pi/n} \qquad (n \text{ is some integer})$$

may be added to the mappings (5.70).

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1) This case takes place when the group G is finite (in distinction from the previous considerations).

REMARK. *In the case when* $mp = 1$, *we are restricted by the above-mentioned general characteristic of the mappings added to the group* Γ.

3) *The group* Γ *consists of the mappings*

$$u^{*1} = u^1 e^{i\theta_1}, \quad u^{*2} = u^2 e^{i\theta_2},$$

to which one more mapping $u^{*1} = u^2$, $u^{*2} = u^1$ *may be added. In this case the group* Γ *depends on two parameters.*

4) *The group* Γ *consists of the mappings*

$$\begin{aligned} u^{*1} &= u^1 e^{i\omega} \cos\varphi - u^2 e^{i\theta} \sin\varphi; \\ u^{*2} &= u^1 e^{-i\omega} \sin\varphi + u^2 e^{-i\theta} \cos\varphi. \end{aligned} \tag{5.71}$$

In this case the group Γ *depends on three parameters.*

5) *The group* Γ *includes all homogeneous affine mappings, with the modulus of their Jacobian* $|\partial u^*/\partial u| = 1$, *that leave the form* $|u^1|^2 + |u^2|^2$ *invariant. In this case the group* Γ *depends on four parameters.*

Thus in the second case the domain $R(D, t)$ turns out to be an (m, p)-circular domain (in particular, one obtains a disk domain for $m = p$ and a semidisk domain if m or p is equal to zero). In the third case $R(D, t)$ is a bicircular domain. *It can be shown that the fourth and fifth cases lead to a hyperball.* [1)]

3. Special cases. It is most interesting to investigate the problem of whether all the domains considered in Theorem 23.2 (i.e., bicircular, semidisk, disk and (m, p)-circular domains) are representative. An affirmative answer to this problem enables us (by the aid of Theorem 22.8) to find all biholomorphic mappings of these domains onto one another.

In his paper dealing with the definition of the concept of a representative domain, Bergman [3] proved the

THEOREM 23.3. *Every bicircular domain*

$$D = \{ |z^2| < R(|z^1|), \quad |z^1| < r \},$$

where $R(|z^1|)$ *is a function integrable in the sense of Lebesgue for* $0 < |z^1| < r$, *is a representative domain relative to its center.*

Immediately after this the following theorem was proved (see Welke [1]):

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1) See H. Cartan [2], (I), p. 78. The fourth case cannot in fact occur. It can be established that if Γ contains the family of transformations (5.71) depending on three parameters, then it also contains other mappings and depends on four parameters.

THEOREM 23.4. *Every disk domain including its center*

$$D = \{|z^1|^2 + |z^2|^2 < \varphi^2(s)\}$$

(*where* $s = z^1 z^{2^{-1}}$; *the boundary of the domain is given by the equation*

$$|z^2| = \varphi(s)(1+s^2)^{-1/2} = R(s),$$

or

$$|z^1| = \varphi\left(\frac{1}{t}\right)(1+t^2)^{-1/2} = R^*(t), \text{ where } t = z^2 z^{1^{-1}};$$

the functions $R(s)$ *for* $|s| < \infty$ *and* $R^*(t)$ *for* $|t| < \infty$ *are assumed integrable in the sense of Lebesgue*) *is a representative domain relative to its center.*

The proof of these theorems is based on the fact that (as can be confirmed by a direct calculation) the functions $u^1(z^1, z^2)$, $u^2(z^1, z^2)$ from formula (5.51), which define the mapping onto the representative domain, may be alternatively defined as

$$u^1 = \frac{f_{010}(z^1, z^2)}{M_0(z^1, z^2)}; \quad u^2 = \frac{f_{001}(z^1, z^2)}{M_0(z^1, z^2)}, \tag{5.72}$$

where f_{010}, f_{001} are functions which minimize the integral $\|f\|_D$ under some supplementary conditions (mentioned in the proof of (1.59)).

For a bicircular domain the functions f_{010}, f_{001}, $M_0(z^1, z^2)$ can be calculated by starting from the possibility of representating all the permissible functions by double power series. After the choice of the first three coefficients of this series (on the basis of the supplementary conditions mentioned for the definition of f_{010}, f_{001}) and of the first coefficient in the series for the function $M_0(z^1, z^2)$, we find (just as in the case of the domain D_2 in the subsequent proof of Theorem 23.4) that

$$f_{010}(z^1, z^2) = z^1, \quad f_{001}(z^1, z^2) = z^2, \quad M_0(z^1, z^2) = 1.$$

Hence Theorem 23.3 is proved.

We turn to the case of the disk domain. We represent the disk domain D involved as the sum of the three domains D_1, D_2, D_3, where $D_1 = \{|z^1| \le r, |z^2| \le r\}$ is a closed bicylinder (r is taken so small that $D_1 \Subset D$), $D_2 = \{(z^1, z^2) \in D; |z^1/z^2| \le 1; |z^2| > r\}$, $D_3 = \{(z^1, z^2) \in D; |z^2/z^1| < 1; |z^1| > r\}$. [1]) Evidently D_2 and D_3 are disk domains with their centers at the origin. Since $|z^2| > r$ for $(z^1, z^2) \in D_2$, by setting $z^1/z^2 = s$, we can express the condition that the point

1) The connectedness of D_2 and D_3 follows from the starlikeness of the domain D.

(z^1, z^2) belongs to the domain D_2 in the form of the inequality $r < |z^2| < R(s)$, while the analogous condition that the point (z^1, z^2) belongs to the domain D_3 can be written in the form $r < |z^1| < R^*(t)$.

We represent the integral as

$$\|f\|_D^2 = \|f\|_{D_1}^2 + \|f\|_{D_2}^2 + \|f\|_{D_3}^2. \tag{5.73}$$

Since $\bar{D}_1$ is a closed bicircular domain, the functions minimizing the integral $\|f\|_{D_1}^2$ will, in view of Theorem 23.3, have the form

$$f_{010}^{(D_1)}(z^1, z^2) = z^1;\quad f_{001}^{(D_2)}(z^1, z^2) = z^2;\quad M_0^{(D_1)}(z^1, z^2) = 1. \tag{5.74}$$

These functions are obtained as solutions of minimum problems for permissible functions $L^2(D_1)$ consisting of functions regular and square-integrable in the domain D_1. But the functions (5.74) belong to the class $L^2(D)$. Therefore these are solutions of the corresponding minimum problems for the domain D as well.

The domain $\bar{D}_2$ is a closed disk domain with its center at the origin of coordinates. Every function holomorphic in the domain D_2 may be represented by the diagonal series

$$\begin{aligned} f(z^1, z^2) &= \sum_{l=0}^{\infty}\left(\sum_{\nu=0}^{l} a_{l,\,l-\nu}(z^1)^{\nu}(z^2)^{l-\nu}\right) \\ &= \sum_{l=0}^{\infty}(z^2)^l\left(\sum_{\nu=0}^{\infty} a_{l,\,l-\nu}s^{\nu}\right) = \sum_{l=0}^{\infty}(z^2)^l\varphi_l(s) \end{aligned} \tag{5.75}$$

(the second equality of (5.75) holds because here $|z^2| \geq r$). The normalization conditions for obtaining the minimum (1.59) in our case are as follows: we must require $\phi_0(s) = 0$, $\phi_1(s) = s$ to obtain f_{010}; $\phi_0(s) = 0$, $\phi_1(s) = 1$ to obtain f_{001} and $\phi_0(s) = 1$ to obtain $M_0(z^1, z^2)$. We consider the sequence of disk domains

$$D_2^{(n)} = \{r < |z^2| < \theta_n R(s)\},$$

where $0 < \theta_n < 1$, $\lim_{n\to\infty}\theta_n = 1$, and therefore $\lim_{n\to\infty} D_2^{(n)} = D_2$. Then we have

$$\begin{aligned} \|f\|_{D_2}^2 &= \lim_{n\to\infty}\|f\|_{D_2^{(n)}}^2 = \lim_{n\to\infty}\sum_{l,\,l'=0}^{\infty}\int_{|s|<1}\int_{r\leqslant|z^2|\leqslant\theta_n R(s)} z^{2l+1}\bar{z}^{2l'+1}\varphi_l(s)\overline{\varphi_{l'}(s)}\,ds\,d\bar{s} \\ &= \sum_{l=0}^{\infty}\int_{|s|<1}(R^{2l+4}(s) - r^{2l+4})\varphi_l(s)\overline{\varphi_l(s)}\,ds\,d\bar{s}. \end{aligned}$$

Here $R(s) \geq r$. Therefore all integrands are positive, and we obtain the minimum value of the integral when all $\phi_l(s)$ are taken equal to zero (this is allowed by the supplementary conditions). Hence it follows that in the class of functions $L^2(D_2)$, the functions (5.74) give the solutions of the corresponding minimum problems. Evidently these are solutions of the corresponding minimum problems for the class of functions $L^2(D)$ as well.

By analogous reasoning for the domain D_3 we find that the functions (5.74) minimize all the three terms of (5.73) and consequently they minimize the integral $\|f\|_D^2$ under proper supplementary conditions.

With this the theorem is proved.

In the above theorems we have restricted ourselves to the cases of complete disks and bicircular domains, which are simplest from the viewpoint of the calculations. However, analogous theorems may also be established for incomplete domains.

4. A summary of the automorphisms of some domains, which we shall now present (without calculations), closes this section.

1. *Automorphisms preserving the center.*[1)] a) First we must note that automorphisms of the hyperball, bicircular and (m, p)-circular domains for $mp > 0$ (so that the disk domains are already included) coincide with the mappings mentioned in Theorem 23.2. As a result, the assumption that all of these domains are representative is an essential one.

b) Automorphisms of a semidisk domain with the symmetry plane $z^1 = 0$ have the form

$$z^{*1} = z^1 f(z^2); \quad z^{*2} = g(z^2). \tag{5.76}$$

c) Automorphisms of Cartan's $(m, -m)$-circular domain have the form

$$z^{*1} = z^1 f(z^1 z^2); \quad z^{*2} = z^2 g(z^1 z^2) \tag{5.77}$$

or

$$z^{*1} = z^2 f(z^1 z^2); \quad z^{*2} = z^1 g(z^1 z^2). \tag{5.78}$$

d) Automorphisms of an (m, p)-circular domain for $mp < 0$ (moreover $m > 0$, $-p = p' > 0$, $p' \neq m$) have the form

$$z^{*1} = z^1 f(z^{1p'} z^{2m}); \quad z^{*2} = z^2 g(z^{1p'} z^{2m}). \tag{5.79}$$

Thus it is evident that (m, p)-circular domains for $mp \leq 0$ are, in general, not

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1) For the derivation of the formulas given below, see H. Cartan [2], (I), p. 91. The paper of H. Cartan [1] deduces automorphisms for some other domains of special type.

representative and can be so only in the case when the corresponding groups (5.76) – (5.79) are found to be linear.

2. *Automorphisms not preserving the center.* In addition to the automorphisms preserving the center, there are several cases in which the domains under consideration have automorphisms which shift the center. For the investigation of such automorphisms, it is most important to consider the problem of finding all the "homogeneous" domains, i.e., domains with a transitive group of automorphisms. The automorphisms of such a group can translate an arbitrary point of the domain to any other point of that domain.

Such domains will be dealt with in the next section. Here we only note that in the space $C^2_{w,z}$ all such domains are biholomorphically mapped either onto the hyperball $K = \{|w|^2 + |z|^2 < 1\}$ or onto the bicylinder $E = \{|w| < 1, |z| < 1\}$. By a calculation which is omitted here we can show that the group of all automorphisms of the hyperball has the form

$$W = e^{i\theta}\frac{(w+\bar{\lambda}z) - \omega(1+\lambda\bar{\lambda}) + \bar{\mu}\sqrt{1-\omega\bar{\omega}(1+\lambda\bar{\lambda})}\,(w\lambda - z)}{\sqrt{(1+\lambda\bar{\lambda})(1+\mu\bar{\mu})}\,[1-(w+\bar{\lambda}z)\,\omega]},$$

$$Z = e^{i\varphi}\frac{\mu(w+\bar{\lambda}z) - \mu\omega(1+\lambda\bar{\lambda}) - \sqrt{1-\omega\bar{\omega}(1+\lambda\bar{\lambda})}\,(w\lambda - z)}{\sqrt{(1+\lambda\bar{\lambda})(1+\mu\bar{\mu})}\,[1-(w+\bar{\lambda}z)\,\bar{\omega}]}. \tag{5.80}$$

The mapping (5.80) carries the point $(\omega, \lambda\omega)$ into the origin of the coordinates. For $\omega = 0$ the group (5.80) is reduced to the subgroup of stability for the origin of coordinates. In formulas (5.80) ω, λ, μ are complex parameters, while θ, ϕ are real parameters; their choice is evidently subject to the condition $|\omega|\sqrt{1+\lambda\bar{\lambda}} < 1$. Thus the group of all automorphisms of the hyperball K depends on eight real parameters. It is easy to see that for $\omega = 0$ in formula (5.80) there remain four (real) essential parameters.

The group of all automorphisms of the bicylinder E depends on six real parameters and is reduced to the two Möbius groups:

$$W = e^{i\theta}\,\frac{w-\alpha}{1-w\bar{\alpha}}; \quad Z = e^{i\varphi}\frac{z-\beta}{1-z\bar{\beta}}.$$

§24. HOMOGENEOUS BOUNDED DOMAINS

1. Introductory facts.

DEFINITION (homogeneous domains). A domain $D \subset C^n$ is said to be homogeneous if the group $G(D)$ of automorphisms of this domain is transitive, i.e., there

exists for any pair of points z_1, $z_2 \in D$ an automorphism $g \in G(D)$ such that $gz_1 = z_2$.

DEFINITION (symmetric domain). A homogeneous domain $D \subset C^n$ is said to be symmetric if for any point $z_0 \in D$ one can find an automorphism $g \in G(D)$ such that: 1) $gz_0 = z_0$, but $gz \neq z$ if the point $z \in D$ differs from the point z_0; 2) $g \circ g = e$, where $e \in G(D)$ is the identity mapping.

E. Cartan, [1] who in 1935 set about a systematic investigation of homogeneous domains, found all the bounded homogeneous domains in the spaces C^2 and C^3. It was shown that in the space C_z^2 any bounded homogeneous domain can be biholomorphically mapped either onto the hyperball $\{|z_1|^2 + |z_2|^2 < 1\}$ or onto the bicylinder $\{|z_1| < 1, |z_2| < 1\}$. In the space C_z^3 any bounded homogeneous domain can be biholomorphically mapped onto one of the following domains:

1) the hyperball $\{|z_1|^2 + |z_2|^2 + |z_3|^2 < 1\}$;

2) the domain $\{|z_1|^2 + |z_2|^2 < 1, |z_3| < 1\}$;

3) the polycylinder $\{|z_1| < 1, |z_2| < 1, |z_3| < 1\}$;

4) a bounded domain which is obtained by a biholomorphic mapping from the domain $\{y_3 > \sqrt{y_1^2 + y_2^2}\}$.[2]

Here, as usual, $z_k = x_k + iy_k$, $k = 1, 2, 3$.

One can show that all of the above-mentioned domains are not only homogeneous, but also symmetric. In this connection E. Cartan raised the problem: does there exist in the space C^n for $n \geq 4$ a bounded homogeneous nonsymmetric domain?

An answer to this problem was given by I. I. Pjateckiĭ-Šapiro, who, in his paper [1] published in 1959, constructed the first example of a bounded homogeneous nonsymmetric domain. It was shown that similar domains exist in the space C^n of all dimensions, beginning with the fourth. Moreover, in every one of the spaces C^4, C^5, C^6 there appears a finite set of biholomorphically inequivalent homogeneous bounded nonsymmetric domains, but already in the space C^7 the set of such domains attains a continuous cardinality. On the other hand, the set of

1) See E. Cartan [1]. Proofs of many propositions in the theory of bounded symmetric domains can be found in the books by Siegel [1], Pjateckiĭ-Šapiro [2] and Hua [1].

2) For $n \geq 2$ the class of unbounded domains $D \subset P_z^n$ which are not mapped biholomorphically onto bounded domains is significantly wider than for $n = 1$. We note that the mapping $z_1^* = (z_1)^{-1}, z_2^* = (z_2)^{-1}$ transforms the bicylinder $\{|z_1| < 1, |z_2| < 1\}$ into the domain $\{|z_1^*| > 1, |z_2^*| > 1\}$, which does not make up the whole exterior of the former domain. On this point see the article of Behnke-Peschl [1].

bounded symmetric domains is finite for any space C^n.

2. Classical domains. E. Cartan [1] gave the classification of all bounded symmetric domains. These domains are divided into equivalence classes with respect to biholomorphic mappings. Every one of such classes can be defined by specifying one domain belonging to it. Moreover, it evidently suffices to consider only the *irreducible classes*, i.e., the classes that are not products of bounded symmetric domains of lower dimensions. Of the many classes of domains considered above, those specified by the following are irreducible: the disk $\{|z_1| < 1\}$ in the space C^1, the hyperballs $\{|z_1|^2 + |z_2|^2 < 1\}$ and $\{|z_1|^2 + |z_2|^2 + |z_3|^2 < 1\}$ in the spaces C^2 and C^3 and the domain $\{y_3 > \sqrt{y_1^2 + y_2^2}\}$ in the space C^3.

In general, as E. Cartan [1] established, there exist six types of classes of irreducible bounded symmetric domains. Domains belonging to four of them are said to be *classical*, since the groups of their automorphisms are classical semisimple Lie groups. Two of these types are singular in the sense that each of them is encountered in the space C^n of only a single dimension n, respectively, for $n = 16$ and $n = 27$.

We turn to the description of the classical domains.

I. Classical domains of the first type exist in the space C^n with arbitrary dimension n; each of them corresponds to one of the possible decompositions into factors $n = pq$, where p, q are natural numbers, $p \geq q > 0$. The corresponding class is represented by the domain

$$\mathfrak{R}_{\mathrm{I}} = \{E_q - Z^*Z > 0\}. \tag{5.81}$$

Here

$$E_q = \begin{pmatrix} 1 & \dots & 0 \\ \cdot & \cdot & \cdot \\ 0 & \dots & 1 \end{pmatrix}, \qquad Z = \begin{pmatrix} z_{11} & \dots & z_{1q} \\ \cdot & \cdot & \cdot \\ z_{p1} & \dots & z_{pq} \end{pmatrix}.$$

Thus E_q is the unit square matrix of order q. Elements of the rectangular $p \times q$ matrix Z are complex numbers $z_{\alpha\beta}$, the coordinates of points of the space C^n. By Z^* in inequality (5.81) we denote the matrix obtained from the matrix Z by:

1) replacing the elements of this matrix by their complex conjugates (such a matrix is usually denoted by $\overline{Z}$);

2) then taking the transpose of the matrix $\overline{Z}$, i.e., interchanging rows and columns (the transpose of a matrix X is usually denoted by X'). Thus $Z^* = \overline{Z}'$.

We shall maintain all of these notations throughout the following exposition.

Inequality (5.81) means, as usual, that for a point $z \in \mathfrak{R}_{\mathrm{I}}$ the hermitian form whose coefficients are the corresponding elements of the matrix (5.81) is positive definite.

Since there evidently exists a one-to-one correspondence between the points $z \in \mathfrak{R}_{\mathrm{I}}$ and the matrices Z satisfying the condition (5.81), it is usually said that the domain $\mathfrak{R}_{\mathrm{I}}$ consists of such matrices. Analogous results also follow for other classical domains.

We note that in some papers the domain $\mathfrak{R}_{\mathrm{I}}$ is still denoted by the symbol $M_{q,p}$ $(p \geq q)$.

II. Classical domains of the second type exist in the space C^n, where $n = p(p+1)/2$; here p is a positive integer.

The corresponding class is represented by the domain

$$\mathfrak{R}_{\mathrm{II}} = \{E_p - Z\bar{Z} > 0\}. \tag{5.82}$$

Here

$$Z = \begin{pmatrix} z_{11} & \dots & z_{1p} \\ \cdot & \cdot & \cdot \\ z_{p1} & \dots & z_{pp} \end{pmatrix},$$

where $z_{\alpha\beta} = z_{\beta\alpha}$. Thus Z is a symmetric square matrix of order p. Its elements are complex numbers $z_{\alpha\beta}$, the coordinates of points of the space C^n. Evidently the largest number of different quantities $z_{\alpha\beta}$ coincides with the dimension of the space C^n. The domain $\mathfrak{R}_{\mathrm{II}}$ is sometimes called the *generalized Siegel disk of degree* p.

III. Classical domains of the third type exist in the space C^n, where $n = p(p-1)/2$; here the integer $p > 1$.

The corresponding class is represented by the domain

$$\mathfrak{R}_{\mathrm{III}} = \{E_p + Z\bar{Z} > 0\}. \tag{5.83}$$

Here

$$Z = \begin{pmatrix} 0 & \dots & z_{1p} \\ \cdot & \cdot & \cdot \\ z_{p1} & \dots & 0 \end{pmatrix},$$

where $z_{\alpha\beta} = -z_{\beta\alpha}$. Thus Z is a skew-symmetric square matrix of order p. Its elements are complex numbers $z_{\alpha\beta}$, the coordinates of points of the space C^n. Evidently the largest number of different quantities $z_{\alpha\beta}$ coincides with the dimension

of the space C^n.

IV. Classical domains of the fourth type exist in the space C^n of any dimension. The corresponding class is represented by the domain

$$\mathfrak{R}_{\mathrm{IV}} = \{|zz'|^2 - 2\bar{z}z' + 1 > 0,\ |zz'| < 1\}. \tag{5.84}$$

Here $z = (z_1 \cdots z_n)$ and correspondingly $\bar{z} = (\bar{z}_1 \cdots \bar{z}_n)$, $z' = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$.

In the book by L. K. Hua [1] one can find explicit expressions for the kernel functions and groups of automorphisms of the domains $\mathfrak{R}_{\mathrm{I}}$, $\mathfrak{R}_{\mathrm{II}}$, $\mathfrak{R}_{\mathrm{III}}$ and $\mathfrak{R}_{\mathrm{IV}}$.

It is evident that the above-mentioned representations of the classes of classical domains may be replaced by others that are equivalent to them in the sense of biholomorphic mappings. Thus, instead of the generalized Siegel disk $K_p = \mathfrak{R}_{\mathrm{II}}$, one may consider the "generalized Siegel half-plane" T_p onto which the domain K_p can be biholomorphically mapped. The domain T_p consists of all symmetric square matrices with elements $z_{\alpha\beta} = x_{\alpha\beta} + iy_{\alpha\beta}$:

$$Z = \begin{pmatrix} z_{11} & \cdots & z_{1p} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ z_{p1} & \cdots & z_{pp} \end{pmatrix} = X + iY = \begin{pmatrix} x_{11} & \cdots & x_{1p} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ x_{p1} & \cdots & x_{pp} \end{pmatrix} + i \begin{pmatrix} y_{11} & \cdots & y_{1p} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ y_{p1} & \cdots & y_{pp} \end{pmatrix},$$

where the matrix $Y > 0$. In other words, T_p is a radial tubular domain (in the book by Pjateckiĭ-Šapiro such a domain is called a Siegel domain of the first kind); its base in the space $R_{p(p+1)/2}$ is the cone consisting of positive definite symmetric matrices of order p.

3. Siegel domain of the first kind. Consider a tubular domain with an affinely homogeneous base. A domain of the space R_n is said to be affinely homogeneous if it possesses a group of affine mappings $y_k \longrightarrow \tilde{y}_k$, $k = 1, \cdots, n$, onto itself (a group of "*real-affine automorphisms*"), which operate transitively in the domain. It is obvious that real mappings come into question here; they are naturally continued into the whole tubular domain as transformations of the form $(x_k + iy_k) \longrightarrow (\tilde{x}_k + i\tilde{y}_k)$, $k = 1, \cdots, n$. Combined with all possible displacements in the direction of the real axis, they become the transitive group of "complex- (or in other words, holomorphically) affine" automorphisms of the original tubular domain. In this way a correspondence is established between "*complex-affinely homogeneous*" tubular domains in the space C^n and "*real-affinely homogeneous*" domains in the space R_n.

The bounded homogeneous domains are the subject of our investigation. Since the tubular domains are by definition unbounded, we are therefore interested in the tubular domains that can be biholomorphically mapped onto bounded domains. We shall call such domains *biholomorphically equivalent to a bounded domain.* We have the

THEOREM 24.1. *In order that a tubular domain with a convex base should be biholomorphically equivalent to a bounded domain, it is necessary and sufficient that its base should not contain entire straight lines.*

PROOF. I. If the base of the domain contains a straight line, then the tubular domain itself contains a one-complex-dimensional analytic plane. Every holomorphic function bounded at all points of this plane will be constant on it, by Liouville's theorem, and cannot separate points of this plane by a mapping. Thus the condition indicated in the theorem is necessary.

II. If the condition of the theorem is fulfilled, the base of the domain can be confined in an n-polygon (n being the dimension of the space) in which none of its sides is a prolongation of any other. Thus the polygon may be carried into an octant of the space R_n by a suitable affine transformation. The image of the given tubular domain under the corresponding transformation in the space C^n will lie in a product of half-planes. This latter domain may be mapped biholomorphically onto a polycylinder. Hence our assertion is proved.

COROLLARY. *A radial tubular domain (alternatively, a Siegel domain of the first kind) whose base is a convex real-affinely homogeneous domain, is affinely homogeneous and biholomorphically equivalent to a bounded domain.*

In §23.4, Chapter IV, (I) we have introduced the concept of the cone conjugate to a given cone $W \subset R_n$ (we place the vertex of the cone W at the origin of coordinates). Now it is fruitful to generalize this concept somewhat. We shall say that the points $(\tilde{y}_1, \cdots, \tilde{y}_n) \in R_n$ satisfying the condition

$$H(y, \tilde{y}) > 0 \quad \text{for all points } y \in \overline{W} \setminus \{0\},$$

where $H(y, \tilde{y}) = \sum_{k,l=1}^n a_{kl} y_k \tilde{y}_l$ is a positive definite quadratic form, make up the cone $\widetilde{W}_H$ conjugate to the cone W relative to the form H. The cone W is said to be *self-conjugate* if the equality $W = \widetilde{W}_H$ holds for the proper choice of the form H. Vinberg [1] gave the classification of affinely homogeneous self-conjugate cones (see also Gindikin [1], (I), Pjateckiĭ-Šapiro [2]). It has been shown that there exist four "classical" types of irreducible affinely homogeneous self-conjugate cones and, in addition, one "singular" cone of this sort in 27-dimensional space.

It can be proved that *the homogeneous radial tubular domain* (*the Siegel domain of the first kind*) *will be symmetric if and only if a self-conjugate cone lies in its base.* If the base of such a domain is non-selfconjugate, for example, the cone $\{y_1y_2 - y_3^2 > 0,\ y_1y_4 - y_5^2 > 0,\ y_1 > 0\} \subset R_5$ (see Vinberg [1]), then the corresponding homogeneous tubular domain can be shown to be nonsymmetric.

4. Siegel domain of the second kind. An arbitrary homogeneous bounded domain is not biholomorphically equivalent to a tubular domain, or, all the more, to a homogeneous Siegel domain of the first kind. The classical domain of the first type $M_{q,p}$ for $p \neq q$ can serve as an example here.

This fact is ascertained by investigating the group $G(M_{q,p})$. It can be shown that for $p \neq q$ this group has no commutative subgroups of real dimension $n = pq$, which at once implies that the domain $M_{q,p}$ cannot be biholomorphically mapped onto a tubular domain of the space C^n (the group of automorphisms of this sort of domain evidently has such a subgroup).

Conversely, the domain $M_{p,p}$ is biholomorphically equivalent to a Siegel domain of the first kind. This latter is defined in the space C^{p^2} as the domain whose base in the space R_{p^2} is the cone of square positive hermitian matrices

$$Z = \begin{pmatrix} z_{11} & \dots & z_{1p} \\ \cdot & \cdot & \cdot \\ z_{p1} & \dots & z_{pp} \end{pmatrix},$$

where $z_{\alpha\beta} = \bar{z}_{\beta\alpha}$.

However, it can be shown that the domain $M_{q,p}$ for $p \neq q$ is biholomorphically equivalent to some complex-affinely homogeneous domain $Z_{q,p}$. This domain is constructed as follows: Let $\{U\}$ be the collection of all complex rectangular $q \times r$ matrices (where $r = p - q$), and let $\{Z\}$ be the collection of all complex square matrices of order q. The domain $Z_{q,p}$ is defined in the space of the pairs of matrices (Z, U) of complex dimension $q(p - q) + q^2 = n$ by the following condition:

$$\text{a point } (Z, U) \in Z_{q,p} \text{ if } Y - UU^* > 0.$$

Here Y is the hermitian matrix of order q with elements $\eta_{\alpha\beta} = (z_{\alpha\beta} - \bar{z}_{\beta\alpha})/2i$, where $z_{\alpha\beta}$ is the corresponding element of the matrix Z. For $p = q$, $r = 0$, the matrix U does not appear and we obtain a domain biholomorphically equivalent to the domain $M_{p,p}$.

In the domain $Z_{q,p}$ we have the transitive group of affine transformations which is generated by the mappings

$$(Z, U) \to (Z + C + 2iUA^* + iAA^*,\ U + A),$$
$$(Z, U) \to (\Lambda Z \Lambda^*,\ \Lambda U). \tag{5.85}$$

Here A is a complex rectangular $q \times r$ matrix, C is a hermitian matrix of order q, and Λ is a nondegenerate square matrix of order q.

As an example we consider the case when $p = n$, $q = 1$. The domain $M_{1,n}$ is the hyperball $\{|z_1|^2 + \cdots + |z_n|^2 < 1\}$. Then the domain $Z_{1,n} = \{y_1 - |z_2|^2 - \cdots \cdots - |z_n|^2 > 0\}$; for $n = 1$ it reduces to the upper half-plane. The mapping $M_{1,n} \longrightarrow Z_{1,n}$ is defined by the correspondences

$$z_1 \to \frac{iz_1 + 1}{1 - z_1}; \qquad z_k \to \frac{iz_k}{1 - z_1} \quad \text{for} \quad k \neq 1.$$

The domain $Z_{q,p}$ is an example of a Siegel domain of the second kind.

DEFINITION (V-hermitian form). In the n-dimensional real space R_n let V be a convex cone containing no entire straight line. A vector-function $F(u, v)$ of pairs of vectors (points) $u, v \in C^m$, whose value belongs to the space C^n, is called a *V-hermitian form* if:

1) $F(u, v) = \overline{F(v, u)}$;

2) $F(\lambda u_1 + \mu u_2, v) = \lambda F(u_1, v) + \mu F(u_2, v)$, where $\lambda, \mu \in C$;

3) $F(u, u) \in \overline{V}$, where $\overline{V}$ is the closure of the cone V;

4) $F(u, u) = 0$ if and only if $u = 0$.

DEFINITION (Siegel domain of the second kind). The domain $S \subset C_\zeta^{n+m}$ consisting of points $\zeta = (z, u)$, where $z \in C_z^n$, $u \in C_u^m$,

$$\operatorname{Im} z - F(u, u) \in V,$$

V being the cone described in the preceding definition and $F(u, v)$ a V-hermitian form, is called the Siegel domain of the second kind corresponding to the cone V (we sometimes speak of its being associated with the cone V).

For $m = 0$ the Siegel domain of the second kind is reduced to the Siegel domain of the first kind. If the cone V represents the collection of all positive definite hermitian matrices and $F(u, v) = uv^*$, then the Siegel domain of the second kind S is reduced to the domain $Z_{q,p}$.

Just as for the Siegel domain of the first kind, it can be shown that the Siegel domain of the second kind is biholomorphically equivalent to a bounded domain (contained in the product of a certain number of balls).

Consider affine mappings of the Siegel domain S of the second kind. First of

all we shall deal with the mappings

$$(z, u) \longrightarrow (z + c + 2iF(u, a) + F(a, a), u + a), \tag{5.86}$$

where $a \in C^m$, $c \in R_n$. These transformations do not yet form a transitive group in the domain S. However, if the cone V has a transitive group of affine automorphisms $\{\Lambda\}$ and there can be put into correspondence with each of them a complex affine mapping $\widetilde{\Lambda}$ of the space C^m onto itself, such that $\Lambda F(u, v) = F(\widetilde{\Lambda}u, \widetilde{\Lambda}v)$, then the correspondence

$$(z, u) \longrightarrow (\Lambda z, \widetilde{\Lambda}u)$$

will be an affine automorphism of the domain S. Then all of these transformations together form a transitive group of automorphisms of the domain S.

We have the

THEOREM 24.2 (Pjateckiĭ-Šapiro [2]). *The Siegel domain of the second kind S is always biholomorphically equivalent to a bounded domain. In order that the domain S should be affinely homogeneous it is necessary and sufficient that the group of automorphisms of the cone V should contain an affine transitive subgroup $G_0(V)$ with the following property: there corresponds to each automorphism $\Lambda \subset G_0(V)$ an affine mapping $\widetilde{\Lambda}$ of the space C^m onto itself such that*

$$\Lambda F(u, v) = F(\widetilde{\Lambda}u, \widetilde{\Lambda}v).$$

The Siegel domain of the second kind $Z_{q,p}$ that we have constructed above was associated with the cone of positive definite hermitian matrices. Now we can write down the totality of Siegel domains of the second kind associated with the cone V of positive definite symmetric matrices $\{Y\}$ of order p. Further, we have $n = p(p + 1)/2$ and the space C^m consists of complex rectangular $p \times s$ matrices (where s is a certain natural number) for which the first t_i elements of the ith row are equal to zero, $t_i \leq t_j$ for $i < j$. We take $F(u, v) = (uv^* + \bar{v}u')/2$.

In this case the group of automorphisms of the cone V contains the affine transitive subgroup $G_0(V)$ which consists of transformations $\Lambda(Y) = \Lambda Y \Lambda'$, where Λ is an upper triangular real matrix, while the transformations $\widetilde{\Lambda}(u)$ with $u \in C^m$ are defined by the equality $\widetilde{\Lambda}(u) = \Lambda u$.

Evidently $\Lambda F(u, v) = F(\widetilde{\Lambda}u, \widetilde{\Lambda}v)$; thus all the Siegel domains of the second kind that we have constructed are affinely homogeneous.

It can be shown that these will be symmetric only for $s = 0$ (then the generalized Siegel half-plane T_p is obtained). If $s \neq 0$, we have homogeneous nonsymmetric domains. The first example of this sort of domain, constructed by

I. I. Pjateckiĭ-Šapiro, corresponds to the values $p = 2$, $s = 1$, $t_1 = 0$, $t_2 = 1$. This domain consists of points $(z_1, z_2, z_3, u) \in C^4$ for which

$$y_1 - |u|^2 > 0, \quad (y_1 - |u|^2)y_2 - y_3^2 > 0.$$

Other examples of Siegel domains of the second kind can be found in the book by Pjateckiĭ-Šapiro [2] and in the article by Gindikin [2], (I).

The role of Siegel domains of the second kind in the theory of biholomorphic mappings is made clear by the following

THEOREM 24.3 (Vinberg, Gindikin, Pjateckiĭ-Šapiro [1]). *Every bounded homogeneous domain $D \subset C^n$ can be mapped biholomorphically onto an affinely homogeneous Siegel domain of the second kind.*

The proof of this theorem is based on the possibility of reducing the problem of the classification of bounded domains of the space C^n to a purely algebraic one. In any domain of the space C^n satisfying the conditions of Theorem 24.3 some solvable Lie group of automorphisms operates transitively. [1]) It can be shown that the Lie algebra for such a group is a j-algebra (see the definition, for example, in the book by Pjateckiĭ-Šapiro [2]). The proof of Theorem 24.3 is reduced to the investigation of the structure of such a j-algebra.

We have noted above that the tubular domain with a real-affinely homogeneous base containing no entire straight lines is itself a complex-affinely homogeneous domain which is biholomorphically equivalent to a homogeneous Siegel domain of the second kind. The latter is always so chosen that the values of the corresponding vector-form $F(u, v)$ belong to R_n when $u, v \in R_m$.

We have the

THEOREM 24.4. *The Siegel domain of the second kind is biholomorphically equivalent to a complex-affinely homogeneous tubular domain $\widetilde{S}$ if and only if after an appropriate linear transformation in the space C^m the values of the form $F(u, v)$ belong to R_n when $u, v \in R_m$.*

Then we can take as $\widetilde{S}$ a domain with the base $\{[y - F((\mathrm{Im}\, u)/i, (\mathrm{Im}\, u)/i)] \in V\}$; the corresponding mapping will be defined by the formulas

$$z \longrightarrow z - iF(u, u), \quad u \longrightarrow \sqrt{2}\, u.$$

DEFINITION (V-bilinear symmetric form). Let V be a convex cone in the

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1) In this case the stationary subgroup for any point of the domain reduces to the identity mapping.

space R_n containing no entire straight lines. A vector-function $F(r, r')$ of a pair of vectors (points) $r, r' \in R_m$, whose values belong to the space R_n, is called a *V-bilinear symmetric form* if: 1) $F(r, r') = F(r', r)$; 2) $F(\lambda r_1 + \mu r_2, r') = \lambda F(r_1, r') + \mu F(r_2, r')$; 3) $F(r, r) \in \bar{V}$, where $\bar{V}$ is the closure of the cone V; 4) $F(r, r) = 0$ if and only if $r = 0$.

REMARK. If a V-hermitian form $F(u, v)$ has the property mentioned in Theorem 24.4, then its restriction on $R_m \subset C^m$ is a V-bilinear symmetric form.

DEFINITION (homogeneous real Siegel domain). A domain $\mathfrak{P}$ of the space R_{n+m} is called a *homogeneous real Siegel domain* if it consists of all points $(y, r) \in R_{n+m}$ for which $y \in R_n$, $r \in R_m$ and $y - F(r, r) \in V$. Here $V \subset R_n$ is a convex affinely homogeneous cone containing no entire lines, and $F(r, r')$ is a V-bilinear symmetric form. In addition, it is assumed that the group of affine automorphisms of the cone V has a transitive subgroup $G_0(V)$ in V such that to each transformation $\Lambda \in G_0(V)$ there corresponds an affine mapping $\tilde{\Lambda}$ of the space R_n onto itself which satisfies the condition $\Lambda F(r, r') = F(\tilde{\Lambda} r, \tilde{\Lambda} r')$.

From Theorem 24.4 there follows the

COROLLARY. *Every affinely homogeneous domain of the space R_{n+m}, containing no entire straight lines, is affinely equivalent to a homogeneous real Siegel domain.* [1)]

REMARK 1. The affine equivalence of two domains is defined analogously to their biholomorphic equivalence; here one has only to consider the affine mapping in place of the holomorphic mapping.

REMARK 2. In the homogeneous real Siegel domain $\mathfrak{P}$ there is the transitive group generated by the affine transformations

$$y \to y + 2F(r, a) + F(a, a), \qquad r \to r + a;$$
$$y \to \Lambda y; \qquad r \to \tilde{\Lambda} r;$$

here a is an arbitrary point of the space R_m.

5. Some special functions. We consider the partition of the space C^{n+m} associated with the Siegel domain of the second kind S (see Pjateckiĭ-Šapiro [4], Gindikin [2], (I)). Starting from an affinely homogeneous convex cone V containing no entire straight lines (in subsections 5 and 6 of the present section, we shall

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1) This result can be obtained by the use of Theorems 24.3 and 24.4 (see Vinberg [2]).

only consider such domains and will not repeat this fact as a rule), we first carry out the partition of the space $R_n \supset V$ and then of the complex space C^n. At its basis there lies the

LEMMA 1. *Let* $\mathfrak{P} = \{y - F(r, r) \in V\} \subset R_{n+m}$ *be a homogeneous real Siegel domain. Then the domain of the space* R_{n+m+1}, *which consists of points* $x = (t, y, r) \in R_{n+m+1}$, *where* $t \in R_1$, $y \in R_n$, $r \in R_m$, *and is defined by the conditions*

$$ty - F(r, r) \in V, \quad t > 0,$$

is a cone of the type in question.

REMARK. The transitive group of affine automorphisms of this cone is generated by the mappings:

$$y \to y + 2F(r, a) + tF(a, a), \qquad r \to r + ta, \qquad t \to t;$$
$$y \to \Lambda y, \qquad r \to \tilde{\Lambda} r, \qquad t \to t;$$
$$y \to y, \qquad r \to \sqrt{\mu}\, r, \qquad t \to \mu t.$$

Here μ is a positive number, while a is an arbitrary point of the space R_m.

Suppose that we are given a certain cone of the class under consideration. We shall take any homogeneous real Siegel domain associated with it, and thus we shall again have the sort of cone that was specified in Lemma 1. Then we shall repeat this construction several times. We have the

THEOREM 24.5. *Any cone* V *of the type in question can be obtained by the construction based on Lemma* 1, *beginning with a one-dimensional cone* (*half-line*), *after a certain number of steps* $l = l(V)$. *Here, we always have* $l < \infty$.

The number $l = l(V)$ is called the *rank* of that cone.

Take a cone $V \subset R_n$ of rank l. Let V^i, $i = 1, \cdots, l$ (where V^1 is a half-line and $V^l = V$) be the cones obtained in the above-mentioned construction, let $\mathfrak{P}^i$ be the homogeneous real Siegel domain generating the cone V^i and let F_i be the corresponding vector-form. In the construction of the cone V^i for the domain $\mathfrak{P}^i$ a one-dimensional space is joined to the space in which the domain $\mathfrak{P}^i$ lies. We denote it by $R_{(i)}$, and the corresponding variable by t_i. Let R^{m_i} be the space on which the vector-form F_i is defined, while its values lie in the space containing the cone V^{i-1}. We consider the components F_i^k of the vector-form F_i lying in the one-dimensional spaces $R_{(k)}$ $(k < i)$ that we have introduced. It can be shown that:

1) F_i^k are scalar bilinear symmetric non-negative forms of the quantities r_s,

r'_s $(1 \leq s \leq \operatorname{Dim} R^{mi})$;

2) the entire space R^{mi} is partitioned into the direct sum of the subspaces $R_{(ik)}$ $(k < i)$, on each of which the corresponding component F_i^k concentrates (i.e., this component vanishes only on the subspaces $R_{(ij)}$ for $j \neq k$). Thus it turns out that the space R_n can be partitioned into the direct sum of the subspaces $R_{(i)}$ $(1 \leq i \leq l)$, $R_{(ik)}$ $(1 \leq k < i)$. This is the *canonical partition of the space R_n associated with the cone V.*[1] In the space R_n we choose a basis of vectors (coordinate vectors) satisfying the following two conditions: 1) the basic vectors belong to the subspaces

$$R_{(1)}, R_{(21)}, R_{(2)}, R_{(31)}, R_{(32)}, R_{(3)}, \cdots, R_{(l)}$$

and are ordered in correspondence with this sequence (within each subsequence $R_{(ik)}$ the order of the basic elements is immaterial); 2) by virtue of the choice of the basic vectors belonging to the subspace $R_{(ik)}$, the form F_i^k is reduced to the sum of squares.

It can be proved that the transitive group $G_0(V)$ of affine automorphisms of the cone V will be expressed by triangular matrices for such a choice of coordinate vectors.

As is shown by direct verification, to each vector (point) $x \in V$ there corresponds, in a one-to-one way, a definite element in the group $G_0(V)$, expressed by a triangular matrix with unity along the principal diagonal, which carries this vector x into a vector $\tilde{x} \in V$, all of whose components that lie in the space $R_{(ik)}$ are equal to zero.

We denote by $\chi_i(x)$ the components of the vector $\tilde{x}$ lying in the space $R_{(i)}$. These are rational functions of the coordinates of the vector x; by means of these the cone V may be given by conditions:

$$\chi_i(x) > 0, \qquad i = 1, \ldots, l.$$

We set

$$\mathrm{sp}\,(x) = \sum_{i=1}^{l} t_i(x), \qquad \chi^{\rho}(x) = \prod_{i=1}^{l} \chi_i^{\rho_i}(x),$$

where $\rho = (\rho_1, \cdots, \rho_l)$, $N_{ik} = \operatorname{Dim} R_{(ik)}$,

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1) Another approach to the partition of the space R_n associated with the cone V and some of its related results can be found in the paper by Vinberg [3].

$$N_k = \sum_{i=1}^{k-1} N_{ki}; \qquad N^k = \sum_{i=k+1}^{l} N_{ik}.$$

Further, we shall need the invariant measure of the volume relative to the group $G_0(V)$.[1] A direct calculation shows that such a volume element is

$$\mu(d\omega_x) = \chi^{\delta}(x)d\omega_x,$$

where $\delta = (\delta_1, \cdots, \delta_l)$; $\delta_k = -[1 + (N^k + N_k)/2]$ and $d\omega_x$ is the Euclidean volume element of the space R_n.

DEFINITION. The function

$$\Gamma_V(\rho) = \int_V \exp(-\operatorname{sp}(x))\chi^{\rho}(x)\mu(d\omega_x),$$

where $\rho = (\rho_1, \cdots, \rho_l)$, is called the *gamma function of the cone* V.

This integral converges absolutely when $\operatorname{Re} \rho_i > N^i/2$, $i = 1, \cdots, l$, but the function $\Gamma_V(\rho)$ can be analytically continued to other values of ρ.

THEOREM 24.6.

$$\Gamma_V(\rho) = \pi^{\frac{n-l}{2}} \prod_{i=1}^{l} \Gamma\left(\rho_i - \frac{N^i}{2}\right),$$

where $\Gamma(\rho)$ *is the classical gamma function.*

Among the point vectors of the space $R_n \supset V$ we introduce a partial ordering associated with the cone V as follows: $x_1 > x_2$, where $x_1, x_2 \in R_n$, if $x_1 - x_2 \in V$. We denote by $[0, \lambda]$, where $\lambda \in V$, the collection of all points x for which $0 < x < \lambda$, i.e., $x \in V, \lambda - x \in V$; and by the symbol $\int_0^{\lambda}$ the integral over that set.

Lastly, we denote by e the point of the cone V for which the coordinates corresponding to the subspaces $R_{(i)}$ are equal to unity, while those corresponding to the subspaces $R_{(ik)}$ are equal to zero.

DEFINITION.[2] The function

$$B_V(\rho, \sigma) = \int_0^e \chi^{\rho}(x)\chi^{\delta+\sigma}(e - x)\mu(d\omega_x),$$

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[1] For the definition of the invariant measure, see, for example, A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Actualités Sci. Ind. No. 869, Hermann, Paris, 1940; Russian transl., IL, Moscow, 1950.

[2] The beta function and the hypergeometric function of the cone V will not be used in the sequel. Here we shall only present some relevant facts and not a complete account.

where $\rho = (\rho_1, \cdots, \rho_l)$, $\sigma = (\sigma_1, \cdots, \sigma_l)$, is called the *beta function of the cone V.*

This integral converges absolutely when

$$\operatorname{Re}\rho_i, \quad \operatorname{Re}\sigma_i > \frac{1}{2} N^i, \qquad i = 1, \ldots, l,$$

but the function $B_V(\rho, \sigma)$ can be analytically continued to other values of ρ and σ.

It is also to be noted that if, instead of e, another point $e_1 \in V$ is taken as the upper limit of the above integral, then the integral is only multiplied by the quantity $\chi^{\rho+\sigma+\delta}(e_1)$.

THEOREM 24.7.

$$B_V(\rho, \sigma) = \frac{\Gamma_V(\rho)\,\Gamma_V(\sigma)}{\Gamma_V(\rho+\sigma)}.$$

DEFINITION. The function

$$F_V(\rho, \sigma, \tau, \lambda) = \frac{1}{B_V(\sigma, \tau-\sigma)\,\chi^{\tau+d}(\lambda)} \int_0^\lambda \chi^\sigma(x)\,\chi^{-\rho}(e-x)\,\chi^{\tau-\sigma+\delta}(\lambda-x)\,\mu(d\omega_x),$$

where $\lambda \in V$, $0 < \lambda < e$, $\rho = (\rho_1, \cdots, \rho_l)$, $\sigma = (\sigma_1, \cdots, \sigma_l)$ and $\tau = (\tau_1, \cdots, \tau_l)$; while $\delta = (\delta_1, \cdots, \delta_l)$ is the same as given above, is called the *hypergeometric function of the cone V.*

This integral converges absolutely when $\operatorname{Re}\rho_i$, $\operatorname{Re}\sigma_i$, $\operatorname{Re}(\tau_i - \sigma_i) > N^i/2$, but can be analytically continued to other values of ρ, σ, τ.

All of these functions, introduced by Gindikin [2], (I), [1] go over, for the case of a half-line, to the classical Euler integrals and the hypergeometric function. But these functions are far from exhausting all the special functions associated with the cone V. It is to be noted that, as in the one-dimensional case, for some values of the parameters ρ, σ, τ the function $F_V(\rho, \sigma, \tau, \lambda)$ reduces to a polynomial. By considering degenerate cases we are led to functions which reduce to Bessel functions for definite values of the parameters. By means of the gamma function of the cone V we can make a natural generalization of the differential operator of a fractional order to the case of several variables.

6. Calculation of the kernel function of an affinely homogeneous Siegel domain of the second kind. Let $S \subset C^{n+m}$ be an affinely homogeneous Siegel domain of the second kind and let $V \subset R_n$ be its associated cone.

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1) For the "classical" cones some of these functions have been investigated earlier by Siegel, Selberg, Pjateckiĭ-Šapiro, Bochner, Herz and other mathematicians.

The canonical partition of the space R_n associated with the cone V naturally generates the partition of the complex space $C^n \supset R_n$ into a sum of subspaces. Consider the space C^m on which a vector-form $F(u, v)$ is defined. It is found that those components of the vector-form $F(u, v)$ that take values lying in the subspaces $R_{(i)}$ of the space R_n are concentrated on the subspaces $C^{q_i}_{(i)}$ of the space C^m $(1 \le i \le l)$, where the space C^m is the direct sum of the spaces $C^{q_i}_{(i)}$. In what follows we shall write $q = (q_1, \cdots, q_l)$.

The partition of the space C^{n+m} thus obtained into the direct sum of the subspaces is called the *canonical partition of the space* C^{n+m} *associated with the Siegel domain of the second kind* S.

Now we can formulate the

THEOREM 24.8 (Gindikin [2], (I)). *The kernel function* $K(\mathfrak{z}, \overline{\zeta})$ *of an affinely homogeneous Siegel domain of the second kind* $S \subset C^{n+m}$, *where* $\mathfrak{z} = (z, u) \in S$, $z \in C^n$, $u \in C^m$ *and* $\zeta = (w, v) \in S$, $w \in C^n$, $v \in C^m$, *is defined by the equation*

$$K(\mathfrak{z}, \overline{\zeta}) = \frac{\Gamma_V(q-2\delta)}{\pi^{m+n}2^{n+l}\Gamma_V(-\delta)} \chi^{2\delta-q}\left(\frac{z-\overline{w}}{2i} - F(u, v)\right).$$

This formula contains as a special case the results of L. K. Hua [2] concerning the kernel functions of the classical domains to which we have referred above.

As is known, the Bergman kernel function plays the role of a reproducing kernel of the integral representation mentioned in Theorem 4.7. In this representation the integration is carried out over the volume of the domain in question. Along with this there exists an integral representation in which the integration is taken over the boundary of the domain or over some part of it (see Chapter IV, (I)). Such integral formulas were constructed for radial tubular domains by S. Bochner and for arbitrary tubular domains by S. G. Gindikin (see §23.4–6, Chapter IV, (I)). These integral representations can be generalized in the cases of the Siegel domain of the second kind which we have considered. The kernels of these integral representations (the so-called Szegö kernels), are expressible in terms of the special functions we have analyzed.

These functions also have other analytical applications.

§25. SOME ESTIMATES FOR BIHOLOMORPHIC MAPPINGS

1. Bounded mappings of domains of the space C^2. The analysis of the metric invariant under biholomorphic mappings allows us to find limits for the change of

quantities of the Euclidean metric (lengths, angles, areas, curvatures of curves, curvatures of surfaces, etc.) under such mappings. We write:

$$dS^2 = dz^1 d\bar{z}^1 + dz^2 d\bar{z}^2.$$

Then, considering the form $ds^2 - \lambda dS^2$ (ds^2 is defined by equality (5.18)), we evidently obtain:

$$\lambda_2^{-1}\, ds^2 \leqslant dS^2 \leqslant \lambda_1^{-1}\, ds^2. \tag{5.87}$$

Here

$$\lambda_1 = \sqrt{\Delta}\, p^{-1}\left(1 - \sqrt{1 - p^2}\right), \quad \lambda_2 = \sqrt{\Delta}\, p^{-1}\left(1 + \sqrt{1 - p^2}\right),$$
$$p = \frac{2\sqrt{\Delta}}{T_{1\bar{1}} + T_{2\bar{2}}},$$

and Δ is the discriminant of the form (5.18).

If a domain D is mapped biholomorphically onto a domain D^* by means of a pair of functions $z^{*k} = z^{*k}(z^1, z^2)$, $k = 1, 2$, a relation similar to (5.87) may be written for the latter domain (we shall write the corresponding quantities for D^* with asterisks). Then we obtain (see Bergman [6])

$$\frac{\lambda_1}{\lambda_2^*} \leqslant \frac{dS^*}{dS} \leqslant \frac{\lambda_2}{\lambda_1^*}. \tag{5.88}$$

Analogously we may also obtain limits for the change of the angles under biholomorphic mappings (see Fuks [1], (I)). (For this purpose we must start from (5.25), (5.26) and (5.88).) Hence, on the basis of formula (5.27), it is also easy to obtain estimates for an ordinary angle between two directions.

Analogously, by comparing the corresponding quantities in the Bergman and Euclidean metrics, we may obtain estimates for the change of the curvature of analytic surfaces (see Fuks [1], (I)) (it is necessary to use formula (5.32)) and for the change of the curvature of hypersurfaces (see Mitrohin [1], Voskresenskiĭ [1]) (it is necessary to consider the quantities (5.33) in both metrics).

As is seen, for example, from inequalities (5.88), the kernel functions (and their derivatives) for the domains D and D^* enter into the estimates obtained. Following Bergman, it is possible to put these estimates into a more easily visualized geometric form. This entails using the fact that the quantities appearing in these estimates may be composed of the solutions of minimal problems (see formula (1.59) analyzed in §4, Chapter I) ("principle of minimal problems"). Then, by forming at the point z in question of the domain D the hyperballs of radii m and M

(where m and M are the smallest and greatest distances from the point z to the boundary of the domain D) and at the point z^* (corresponding to z) the hyperballs of radii m^* and M^*, and by applying relation (1.63) (in effect, reinforcing the corresponding inequalities), we can, on the basis of formula (1.54), replace all quantities entering into these inequalities by m, M, m^* and M^*. In this way, using inequalities (5.26), we obtain, for example, for the change of the angles F the estimates

$$\frac{m^{*3}\left(M^3-\sqrt{M^6-m^6}\right)}{m^3\left(M^{*3}-\sqrt{M^{*6}-m^{*6}}\right)}\sin F\leqslant\sin F^*$$
$$\leqslant\frac{m^3\left(M^{*3}-\sqrt{M^{*6}-m^{*6}}\right)}{m^{*3}\left(M^3-\sqrt{M^6-m^6}\right)}\sin F.$$

We shall not dwell on other analogous inequalities (see, for example, the paper by Šmatkov [1], where estimates are given for the changes of m-volumes under biholomorphic mappings).

2. **Generalizations of Schwarz's lemma.** We now consider the estimates valid for biholomorphic mappings which transform the domain D into a domain G lying inside the domain D. In the case of conformal mappings the fundamental fact in this connection is Schwarz's lemma, which in its invariant form (stated first by Pick) asserts:

If a function $w = w(z)$ maps a domain $D \subset C^1$ onto a domain $G \subset C^1$, where $D \supset G$, then for the length $ds^2 = K|dz|^2$ of one and the same element, measured by means of invariant metrics in the spaces D and G, we have the relation

$$ds_D^2 \le ds_G^2. \tag{5.89}$$

Inequality (5.89) is easily obtained by applying relation (1.64) to the kernel function appropriate for domains of any connectedness. Evidently this method is applicable for getting the corresponding estimates in the theory of biholomorphic mappings. [1)]

Thus, consider the metric (5.20) in the domain $D \subset C_z^2$. Take its invariant $I = \sqrt{K/\Delta}$, where $\Delta = T_{1\bar{1}}T_{2\bar{2}} - T_{1\bar{2}}\,T_{2\bar{1}}$. Then the four-dimensional volume element will evidently be equal to

$$d\Omega^{(4)} = \sqrt{g}\,d\omega = K\,d\omega;$$

here g is the discriminant of the form (5.20) and $d\omega$ the volume element of the

1) Bergman [7]. See Fuks [4] for another way to obtain the inequalities which are generalizations of inequalities (5.89).

four-dimensional Euclidean space.

For the case of a two-dimensional surface we shall start from the definition of the analytic area confined between vectors u^{α}, v^{α}, namely, the quantity

$$\sigma = \frac{1}{2} | u^{\alpha} | \cdot | v^{\alpha} | \sin \theta,$$

where θ is the first analytic angle between these vectors. From formula (5.26) it follows immediately that $\sin \theta$ is the same for all metrics (5.20) with any l. By again putting $l = \sqrt{K/\Delta}$, we have $\sigma = \sqrt{K}\ |u^1 v^2 - u^2 v^1|$, or

$$d\Omega^{(2)} = \sqrt{K} \left| \frac{\partial (z^1, z^2)}{\partial (\alpha^1, \alpha^2)} \right| d\alpha^1 \, d\alpha^2$$

for the surface element (if the surface is given by the functions $z^k = z^k(\alpha^1, \alpha^2)$ of real parameters (α^1, α^2)).

By observing that the quotient of the above-mentioned quantities divided by the corresponding Euclidean ones are solutions of the minimal problems, we obtain the following theorem:

THEOREM 25.1. *If $G \subset D$, then*

$$d\Omega_D^{(s)} \leqslant d\Omega_G^{(s)}; \qquad s = 2,\ 4. \tag{5.90}$$

The inequality of the form (5.90) for $s = 4$ may be interpreted as follows: the hyperellipsoid $I_{(z)}$:

$$\left\{ \sqrt{\frac{K}{\Delta}} \left[T_{m\bar{n}}(z) Z^m \bar{Z}^n \right] < 1 \right\}. \tag{5.91}$$

is the indicatrix of the metric under consideration at the point $z \in G$ (see the definition of the indicatrix in §21.1). The volume $V(I_{(z)})$ of the hyperellipsoid (5.91) is equal to $\pi^2 (2K)^{-1}$ (as is seen immediately). Therefore if $G \subset D$, then for the indicatrix of the metric (5.20) with $l = \sqrt{K/\Delta}$ we have the inequality

$$V\left(I^G_{(z)}\right) \leqslant V\left(I^D_{(z)}\right).$$

In the theory of conformal mappings one obtains from inequality (5.89) a series of corollaries concerning mappings onto a part of the domain in question, mappings giving rise to three values and so forth.

Analogous corollaries may also be obtained in the case of biholomorphic mappings. For example, we can derive the following inequality: Suppose that we are given the biholomorphic mapping

$$w^k = w^k (z^1, z^2), \qquad k = 1,\ 2, \tag{5.92}$$

of the unit bicylinder D onto the domain G, which gives rise to the planes $w^1 = 0, 1, \infty$ and $w^2 = 0, 1, \infty$. Let $\zeta(z)$, $\zeta(0) = 1$, be the modular function which maps the unit disk onto an infinitely-sheeted Riemann surface with branch points $\zeta = 0, 1, \infty$, and the domain A be obtained from D by means of the mapping

$$\zeta^1 = \zeta(z^1), \quad \zeta^2 = \zeta(z^2), \quad \zeta(0) = w_0^1 = w_0^2.$$

$G \subset A$ since the domain $G \subset C_w^2$ gives rise to the above-mentioned planes. Therefore in view of inequality (5.90) we have at the point (w_0^1, w_0^2),

$$d\Omega_A^{(4)}(w_0^1, w_0^2) \leqslant d\Omega_G^{(4)}(w_0^1, w_0^2).$$

We take the point $(w_0^1, w_0^2) \in G$; for the last mapping it corresponds to the point $(0, 0) \in D$; because of the normalization of the mapping we have

$$d\Omega_A^{(4)}(w_0^1, w_0^2) = d\omega(0, 0),$$

where $d\omega(0, 0)$ is the Euclidean volume element in the space C_z^2. For $d\Omega_G^{(4)}(w_0^1, w_0^2)$ we have

$$d\Omega_G^{(4)}(w_0^1, w_0^2) = K_D\, d\omega(z^1, z^2),$$

where $d\omega(z^1, z^2)$ is the Euclidean volume element corresponding to the element $d\omega(w_0^1, w_0^2)$ under the mapping inverse to the mapping (5.92). Consequently we obtain

$$d\omega(0, 0) \leqslant K_D\, d\omega(z^1, z^2),$$

which implies

$$\frac{d\omega(w_0^1, w_0^2)}{d\omega(0, 0)} \geqslant \frac{1}{K_D} \frac{d\omega(w_0^1, w_0^2)}{d\omega(z^1, z^2)}.$$

But

$$\frac{d\omega(w_0^1, w_0^2)}{d\omega(0, 0)} = \left| \frac{\partial(\zeta^1, \zeta^2)}{\partial(z^1, z^2)} \right|^2_{z^k=0}; \quad \frac{d\omega(w_0^1, w_0^2)}{d\omega(z^1, z^2)} = \left| \frac{\partial(w^1, w^2)}{\partial(z^1, z^2)} \right|^2_{z^k=0},$$

hence we finally obtain

$$\frac{1}{\sqrt{K_D}} \left| \frac{\partial(w^1, w^2)}{\partial(z^1, z^2)} \right|_{z^k=0} \leqslant \left| \frac{\partial(\zeta^1, \zeta^2)}{\partial(z^1, z^2)} \right|_{z^k=0}.$$

Thus we have obtained an estimate for the Jacobian of the functions (5.92) which biholomorphically maps the bicylinder and produces in it the above-mentioned analytic planes.

For many problems it is important to have estimates not only for the volume or surface element, but also for the line element ds. It can be shown that in this last

case the situation is somewhat more complicated. For the case of one variable, inequality (5.89) is evidently equivalent to the statement that the indicatrix of the invariant metric $ds^2 = K|dz|^2$ (the disk of radius $1/K(z, \bar{z})$) for the domain G lies inside the corresponding indicatrix for the domain D or, what is the same thing, it has an area smaller than that of the latter. In the case of the space of two or more complex variables the corresponding indicatrix is a hyperellipsoid and, of course, it does not follow from inequality (5.91) that the indicatrices of the corresponding domains lie one inside another, which would be equivalent to inequality (5.89). In addition, it is possible to construct an example of domains D and G, $G \subset D$, such that for a proper choice of the vector dz^k inequality (5.89) does not hold. However one can state the

THEOREM 25.2. *Let a domain $D \subset C^n$ be bounded and strictly analytically convex in the sense of Hartogs; its boundary belongs to the class $\mathcal{C}^2$. Then, if a domain $G \subset D$ is obtained from the domain D by a biholomorphic mapping, at all points $z \in G$ we have*

$$ds_D \leq k\, ds_G. \tag{5.93}$$

Here k is a constant depending only on the domain D.

We shall not reproduce the proof of this theorem (see Bergman [7] for $n = 2$).

C. H. Look [1], [2] proved the following

THEOREM 25.3. *Let a domain $D \subset C^n$ be bounded and homogeneous, and let a domain $G \subset D$ be obtained from the domain D by a biholomorphic mapping. Then at all points $z \in G$ the inequality* (5.93) *is valid, with a constant k depending only on the domain D.*

If inequality (5.93) *can be replaced by the equality $ds_D = ds_G$ at least at one point $z \in G$, then the domain G coincides with the domain D.*

C. H. Look calculated the *minimum* value of the constant k (this quantity is sometimes called the *Schwarz constant*) for classical domains. It was shown that $k_{\min} = 1$ only if the domain D is a hyperball. In this case Schwarz's lemma turns out to be valid in its usual formulation.

The Schwarz constant $k_{\min} = \sqrt{q}$ for the classical domains of the first type $\Re_{\mathrm{I}} = M_{q,p}$ $(p \geq q)$; $k_{\min} = \sqrt{p}$ for the classical domains of the second type $\Re_{\mathrm{II}}$; $k_{\min} = \sqrt{[p/2]}$ for the classical domains of the third type $\Re_{\mathrm{III}}$, where $[p/2]$ is the integral part of the number $p/2$. For the classical domains of the fourth type $\Re_{\mathrm{IV}}$ the Schwarz constant $k_{\min} = \sqrt{2}$ if $n \geq 2$.

It is also noted that the Schwarz constant $k_{\min} = \sqrt{n}$ for the polycylinder $\{|z^1| < 1, \cdots, |z^n| < 1\}$.

3. **Normalized mappings.** Let a domain $D \subset C^2$ be bounded and let a point $c \in D$. Consider the biholomorphic mapping of the closed domain $\bar{D}$ onto some closed domain $\bar{D}^* \subset C^2$

$$z^{*k} = z^{*k}(z), \quad k = 1, 2,$$

which is *normalized* at the point $c = c(c^1, c^2)$ by the conditions

$$z^{*k}(c) = 0, \qquad k = 1, 2, \quad \left|\frac{\partial z^*}{\partial z}\right|_{z=c} = 1. \tag{5.94}$$

Let a point $\zeta \in \partial D$. Assume that the boundary ∂D in some neighborhood of the point ζ is a portion of a hypersurface $\{\Phi = 0\}$ of the class $\mathcal{C}^2$ ($\Phi < 0$ at points of the domain D) and that ζ is an ordinary point. We place the origin of coordinates at the point ζ and give the direction of the coordinate axes in such a way that the equation of the tangent hyperplane at the point ζ to the hypersurface $\{\Phi = 0\}$ takes the form $\operatorname{Re}(z^2) = 0$ and the inequality $\pi/2 < \arg c^2 < \pi$ will be satisfied. Then $-\operatorname{Re}(c^2) = \gamma > 0$ is the distance from the point $c \in D$ to the hyperplane $\operatorname{Re}(z^2) = 0$.

DEFINITION (domain analytically convex with degree b). A domain D is said to be *analytically convex with degree* b at its boundary point ζ, if all hypersurfaces

$$\operatorname{Re}(z^2 + \alpha(z^1)^2) = 0 \tag{5.95}$$

for $|\alpha - 2a| < b$ lie outside this domain, and moreover the number $b > 0$ cannot be taken larger here; α is a complex parameter and $a = (\partial^2\Phi/\partial(z^2)^2)_{z=\zeta}$.

It is obvious that $b \le L(\Phi)$, where $L(\Phi)$ is the value of Levi's determinant at the point ζ. This inequality follows from the Remark 1 stated in §12.3, Chapter II, (I).

We subject the domain D to the mapping:

$$w = z^1 - \frac{c^1}{c^2} z^2, \quad z = z^2 \tag{5.96}$$

($c^2 \ne 0$ because of the above-stated assumption about the domain D). Then the domain D goes over into the domain D^* and the point c into the point $c^*(0, c^2)$. The coordinates of the point ζ and the equation of the tangent hyperplane to the boundary at the point ζ remain unchanged. The Jacobian of the mapping (5.96) is equal to unity everywhere; therefore

$$K_D(c^1, \bar{c}^1, c^2, \bar{c}^2) = K_{D^*}(0, 0, c^2, \bar{c}^2) = K_c.$$

In what follows the domain D^* will be said to be *reduced* for the domain D and the points c and ζ. The following theorem will now be proved.

THEOREM 25.4. *If $b > 0$ is the degree of analytic convexity at the point ζ of the reduced domain D^*, and γ is the distance of the point c to the hyperplane tangent to the boundary ∂D at the point ζ, then*

$$\frac{b}{2\pi^2\gamma^3} \leqslant K_c. \tag{5.97}$$

We note that K_c is an invariant of the biholomorphic mappings normalized by the conditions (5.94). Therefore inequality (5.97) establishes the connection between the change of the quantities b and γ under such mappings.

PROOF. Consider the domain of the space $C^2_{w,z}$:

$$E' = \{\operatorname{Re}(z+2aw^2) > bw\bar{w}\}, \quad E_1 = \{\,|\operatorname{Re}(z+2aw^2)| < bw\bar{w}\},$$
$$E = \{\operatorname{Re}(z+2aw^2) < -bw\bar{w}\}.$$

We prove that $D^* \subset E$. To this end we first show that the points of the domain D^* cannot belong to the domain E_1. In fact, if a point $(w_0, z_0) \in E_1$, then

$$\operatorname{Re}(z_0 + 2aw_0^2) = \theta b\,|w_0|^2 = \operatorname{Re}(\theta b e^{-2i\arg w_0} w_0^2).$$

Here $-1 < \theta < 1$. Hence it follows that

$$\operatorname{Re}\left[z_0 + (2a - \theta b e^{-2i\arg w_0})\, w_0^2\right] = 0.$$

This means that the point (w_0, z_0) lies on the hypersurface from the family (5.95) with the parameter $\alpha = 2a - \theta b e^{-2i\arg w_0}$; it cannot be a point of the domain D^* in view of the assumption on the analytic convexity of that domain.

It is further evident that the domains E', E_1, together with their boundaries, exhaust the entire space $C^2_{w,z}$. Finally it is to be noted that the boundaries of the domains E' and E cannot have three-dimensional parts in common. At common points of the closed domains $\bar{E}'$ and $\bar{E}$, the following inequalities must hold simultaneously

$$\operatorname{Re}(z+2aw^2) \geqslant bw\bar{w},$$
$$-\operatorname{Re}(z+2aw^2) \geqslant bw\bar{w}.$$

Hence it follows that $bw\bar{w} = 0$, or simply $w = 0$ at these points. In this way all such points must lie in a two-dimensional plane.

Now it is clear that, because of the connectedness of the domain D^*, either $D^* \subset E'$, or else $D^* \subset E$. But the point $c \in E$; therefore $D^* \subset E$.

Hence, from formula (1.63) it follows that

$$K_c = K_{D^*}(0,\ 0,\ c^2,\ \overline{c^2}) \geqslant K_E(0,\ 0,\ c^2,\ \overline{c^2}). \tag{5.98}$$

To calculate the quantity $K_E(0, 0, c^2, \overline{c}^2)$ we subject the domain E to the mapping

$$w = \frac{\omega}{\sqrt{b}(\zeta+1)};\quad z + 2aw^2 = \frac{\zeta-1}{\zeta+1}. \tag{5.99}$$

The mapping (5.99) is biholomorphic in the entire space except at the points of the surface $\{z + 2aw^2 = 1\}$. This mapping carries the domain E into the hyperball

$$\omega\bar{\omega} + \zeta\bar{\zeta} < 1,$$

and the point $(0, c^2)$ into the point $(0, (c^2 - 1)/(c^2 + 1))$. Then, by using expression (1.54) for the kernel function of the hyperball and the law of transformation of the kernel function under a biholomorphic mapping, we find that

$$K_E(0,\ 0,\ c^2,\ \overline{c^2}) = \frac{b}{2\pi^2\gamma^3}.$$

Substituting this value into formula (5.98), we obtain inequality (5.97). With this the theorem is proved.

4. Estimates for the approach to the boundary. It is possible to mention several more properties and applications of the invariant metric. Here, first of all, one should note a series of facts characterizing the behavior of the form (5.18), which defines the Bergman metric, and its associated invariants on the boundary of a bounded domain $D \subset C^2$. It was shown that, just as for the kernel function of the domain D, the limiting values of the above-mentioned quantities on the approach of a point $z \in D$ to a point $\zeta \in \partial D$ depend on the structure of the boundary of the domain D and on the method of approach of the point z to the point ζ. We shall present (without proof) one of the relevant results. It is directly relevant to Theorem 5.10 (which deals with the behavior of the kernel function of a domain on the boundary).

THEOREM 25.5 (Bergman [5], Fuks [1]). *Let D be a bounded domain of the space C^2, for which*

1) *the boundary ∂D is a hypersurface of the class $\mathcal{C}^2$;*

2) *at a point $\zeta \in \partial D$ the hypersurface ∂D is convex in the sense of Levi and one of the analytic surfaces passing through that point ζ lies entirely outside the domain D;*

3) *a point* z *approaches the point* ζ, *remaining outside the cone formed by the rays which emanate from the origin of coordinates and make an angle not larger than* α *with the outer normal* N *to the hypersurface* ∂D. *Here* α *is a number satisfying the inequality* $0 < \alpha < \pi/2$.

Then

$$\lim_{z\to\zeta} n^2\left(T_{l\overline{m}}\, u^l\, \overline{u}^m\right) = B, \qquad \lim_{z\to\zeta} R = -\frac{3}{2}.$$

Here n *is the projection of the segment* $z\zeta$ *on the normal* N, *the number* B *is a precisely determined constant,* u^l *is a constant vector and* R *is the Riemann curvature of the Bergman metric in the domain* D *with respect to one-complex-dimensional analytic planes.*

Just as for the kernel function of the domain (see §5.5, Chapter I), there are other limiting relations which hold for quantities connected with the Bergman metric if $L(\partial D)\,|_\zeta = 0$, or if the hypersurface ∂D is shown to be convex in the sense of Levi from within the domain D. In the first case the quantity $T_{l\overline{m}}u^l\overline{u}^m$ increases indefinitely as $z \longrightarrow \zeta$, but the order of increase at infinity turns out to be different. The equality $\lim_{z\to\zeta} R = -3/2$ holds as before in this case. However, here one considers a way of letting the point z approach the point ζ which is different from that in Theorem 25.5.

If $L(\partial D)\,|_\zeta < 0$, the quantity $T_{l\overline{m}}u^l\overline{u}^m$ remains finite as $z \longrightarrow \zeta$.

From the results of §5, Chapter I concerning the continuation of the kernel function of the domain $D \subset C^n$, it can be concluded that the following (see Bergman [2], (I), Sommer-Mehring [1], (I)) theorem holds.

THEOREM 25.6. *Let* $H(D)$ *be the holomorphy hull of a bounded domain* $D \subset C^n$, *then:*

1) *The components* $T^{(D)}_{l\overline{m}}$ *of the metric tensor of the Bergman metric* (5.18), *constructed for the domain* D, *are continued as real-analytic functions into the domain* $H(D)$. *Equality* (5.19) *remains valid in the domain* $H(D)$.

2) *The form* $T^{(D)}_{l\overline{m}}\, dz^l d\overline{z}^m$ *defines in the domain* $H(D)$ *the Kähler metric invariant under biholomorphic mappings of the domain* $H(D)$.

Moreover, it should be noted that, in general, this continued metric does not coincide with the Bergman metric constructed for the domain $H(D)$.

5. The characterization of holomorphically complete complex manifolds by means of a complete Kähler metric was given by H. Grauert [1]. We shall present

his results without proof.

THEOREM 25.7. *On any holomorphically complete complex manifold there exists a complete Kähler metric.*

The following more general theorem also holds.

THEOREM 25.8. *There exists a complete Kähler metric on every complete manifold which can be obtained from a holomorphically complete complex manifold by removing some analytic manifold.*

Theorem 25.8 implies that there may exist a complete Kähler metric on a complex manifold which is not holomorphically complete; for example, it can be defined in the domain $C^n \setminus \{0\}$, which for $n > 1$ is not a domain of holomorphy.

Moreover, it can be shown that for domains over the space C^n without interior branch points and with smooth boundaries the following theorem holds.

THEOREM 25.9. *A domain D over the space C^n without interior branch points and with a smooth boundary is a domain of holomorphy if and only if a complete Kähler metric can be defined in it.*

§26. QUASI-BIHOLOMORPHIC MAPPINGS

In this section some families of mappings of the class $\mathcal{C}^1$ are considered. Each of them contains the collection of biholomorphic mappings and preserves some of their properties, or has properties similar to certain properties of the biholomorphic mappings.

1. Mappings preserving the analytic character of some planes. We shall confine ourselves to the case of two variables w, z and consider the mapping

$$(T) \qquad W = W(w, z), \quad Z = Z(w, z) \tag{5.100}$$

which belongs to the class $\mathcal{C}^1$ in a neighborhood of some point (w_0, z_0). We further take

$$w_0 = z_0 = W(w_0, z_0) = Z(w_0, z_0) = 0,$$

this does not affect the generality of our discussion. Let

$$W = aw + bz + p\bar{w} + q\bar{z}, \quad Z = cw + dz + r\bar{w} + t\bar{z}, \tag{5.101}$$

be the differential of the mapping (5.100). Here

$$a = W'_w(w_0, z_0), \quad b = W'_z(w_0, z_0), \quad p = W'_{\bar{w}}(w_0, z_0),$$
$$q = W'_{\bar{z}}(w_0, z_0), \quad c = Z'_w(w_0, z_0), \quad d = Z'_z(w_0, z_0),$$
$$r = Z'_{\bar{w}}(w_0, z_0), \quad t = Z'_{\bar{z}}(w_0, z_0).$$

Suppose that $J_0 = J(w_0, z_0) \neq 0$, where $J(w, z) = \partial(W, Z, \overline{W}, \overline{Z})/\partial(w, z, \overline{w}, \overline{z})$. Then equality (5.100) defines a homeomorphic, continuously differentiable, mapping of some neighborhood of the point (w_0, z_0) onto a certain neighborhood of the point (W_0, Z_0). We shall say that the mapping (5.100) is monogenic at the point (w_0, z_0) if $p = q = r = t = 0$.

The mapping (5.101) carries an analytic plane $w = \omega z$, taken arbitrarily, into the plane defined by the equation

$$W = \frac{(a\omega + b)\,\overline{(c\omega + d)} - (p\overline{\omega} + q)\,\overline{(r\overline{\omega} + t)}}{|c\omega + d|^2 - |r\overline{\omega} + t|^2}\, Z + \frac{(c\omega + d)\,(p\overline{\omega} + t) - (a\omega + b)\,(r\overline{\omega} + t)}{|c\omega + d|^2 - |r\overline{\omega} + t|^2}\, \overline{Z} \tag{5.102}$$

(and its complex conjugate). This plane can be shown to be analytic if and only if the parameter ω satisfies the equation

$$(a\omega + b)(r\overline{\omega} + t) - (c\omega + d)(p\overline{\omega} + q) = \begin{vmatrix} a & p \\ c & r \end{vmatrix} \omega\overline{\omega} + \begin{vmatrix} a & q \\ c & t \end{vmatrix} \omega + \begin{vmatrix} b & p \\ d & r \end{vmatrix} \overline{\omega} + \begin{vmatrix} b & q \\ d & t \end{vmatrix} = 0. \tag{5.103}$$

In this case the analytic plane $w = \omega z$ under the mapping (5.101) goes over into the analytic plane $W = \Omega Z$, where $\Omega = (a\omega + b)/(c\omega + d)$. Under the mapping (5.100) a surface tangent to such an analytic plane $w = \omega z$ at the origin of coordinates goes over into a surface tangent to the analytic plane $W = \Omega Z$ at the origin of coordinates. Here and in similar cases we shall say that *the mapping* (5.100) *preserves the analytic character of the plane* ω.

By considering equation (5.103) we are led to the following result:

THEOREM 26.1 (Fuks [5]). 1) *If*

$$\{|ad - bc| - |pt - rq|\}^2 > J_0,$$

then there exist two and only two planes ω *preserving their analytic character under the mapping* (5.100).

2) *If*

$$\{|ad - bc| - |pt - qr|\}^2 < J_0 < \{|ad - bc| + |pt - qr|\}^2,$$

then there does not exist any analytic plane ω *preserving its analytic character under the mapping* (5.100).

3) *If*

$$J_0 = \{ | ad - bc | \pm | pt - qr | \}^2,$$

then either there does not exist any plane ω, *or else, on the contrary, there exist an infinite set of planes* ω, *preserving their analytic character under the mapping* (5.100).

4) *The case*

$$\{ |ad - bc| + |pt - qr| \}^2 < J_0$$

is impossible.

It is noted that in view of 1) of the present theorem, every mapping (5.100) with a positive Jacobian J_0 preserves the analytic property of two planes ω.

We shall turn to the discussion on the mappings (5.100) which, by 3) of Theorem 26.1, preserve the analytic character of more than two analytic planes ω. For this case it can be shown that the following theorem holds.

THEOREM 26.2 (Fuks [5]). *The mapping* (5.100) *preserving the analytic character of three planes* ω_1, ω_2, ω_3 *preserves the analytic character of all planes* ω *for which the numbers of* ω *are affixes of the points of a circle* (*on a Riemann sphere of the complex variable* ω) *passing through the points with affixes* ω_1, ω_2, ω_3. *In this case*

$$| at - qc | = | br - dp |, \quad \arg \begin{vmatrix} a & q \\ c & t \end{vmatrix} \cdot \begin{vmatrix} b & p \\ d & r \end{vmatrix} = 2 \arg \begin{vmatrix} a & p \\ c & r \end{vmatrix},$$

$$\arg \begin{vmatrix} a & p \\ c & r \end{vmatrix} = \arg \begin{vmatrix} b & q \\ d & t \end{vmatrix} + \pi. \tag{5.104}$$

Here the last (*third*) *relation can be replaced by the conditions*

$$\arg \begin{vmatrix} a & p \\ c & r \end{vmatrix} = \arg \begin{vmatrix} b & q \\ d & t \end{vmatrix},$$

$$| ar - cp | \cdot | bt - dq | < | at - cq | \cdot | br - dp |. \tag{5.104$_1$}$$

Let a point g of the Riemann sphere of the complex variable ω be the center of the circle considered in Theorem 26.2. (In this connection, it is assumed that the length of the arc of the great circle running from the point g to the above circle does not exceed one quarter of the length of the great circle.) Relations (5.104) and (5.104$_1$) are considerably simplified if, by an analytic rotation of the coordinate system, we make the coordinate plane $w = 0$ coincide with the plane $w = gz$. To this end in formulas (5.101) we put

$$w = \frac{w_1 + gz_1}{\sqrt{1+|g|^2}}, \quad z = \frac{z_1 - \bar{g}w_1}{\sqrt{1+|g|^2}},$$
$$W = \frac{W_1 + gZ_1}{\sqrt{1+|g|^2}}, \quad Z = \frac{Z_1 - \bar{g}W_1}{\sqrt{1+|g|^2}}$$

and take into account relations (5.104) and (5.104_1). Then in the new coordinates the transformations (5.101) for the case at hand take the form

$$\begin{aligned} W_1 &= a_1 w_1 + b_1 z_1 + \mu e^{i\varepsilon} b_1 \bar{w}_1 + \nu e^{i\varepsilon} a_1 \bar{z}_1, \\ Z_1 &= c_1 w_1 + d_1 z_1 + \mu e^{i\varepsilon} d_1 \bar{w}_1 + \nu e^{i\varepsilon} c_1 \bar{z}_1. \end{aligned} \tag{5.105}$$

Here ν and μ are non-negative numbers and ϵ is a real number. In addition ν and μ can vanish simultaneously (in this last case the mapping (5.100) is monogenic at the origin of coordinates and preserves the analytic character of all planes ω), and $\nu \le \mu$, $\nu\mu \ne 1$, $a_1 d_1 - b_1 c_1 \ne 0$ (the last two relations follow from the fact that in the case under consideration $J_0 = (1-\nu\mu)^2 \cdot |a_1 d_1 - b_1 c_1|^2$).

The study of the properties of the mapping (5.105) allows us to supplement Theorem 26.2 by the following proposition:

THEOREM 26.3 (Fuks [5]). *If the mapping* (5.100) *preserves the analytic character of three planes* $\omega_1, \omega_2, \omega_3$, *then after an appropriate analytic rotation of the coordinate system its differential can be put in the form* (5.105). *This mapping (considered in the new coordinates) preserves the analytic character of all of the planes* $\omega = \sqrt{\mu/\nu}\, e^{i\psi}$ *(and only these planes). Here* ψ *is an arbitrary real number.*

Under the mapping (5.105) *the circles with center at the origin of coordinates and lying in the planes* $\omega = \sqrt{\mu/\nu}\, e^{i\psi}$ *go over into ellipses with eccentricity* $2\sqrt[4]{\mu\nu}/(1+\sqrt{\mu\nu})$.

The angles of the second rotation (see §7.3, Chapter I, (I)) for vectors with initial points at the point (0, 0) and lying on one and the same plane $\omega = \sqrt{\mu/\nu}\, e^{i\psi}$ are, in general, not identical if $\mu^2 + \nu^2 \ne 0$. For each of the above-mentioned planes the largest variation of the angle between two such vectors is equal to $2 \arcsin \sqrt{\mu\nu}$ for $\mu\nu < 1$ and to π for $\mu\nu > 1$.

From the last theorem it can be further concluded that the mapping (5.100) will be monogenic at the point (w_0, z_0) if one of the following conditions is satisfied:

(K') The mapping (5.100) preserves the analytic character of the three planes $\omega_1, \omega_2, \omega_3$, and moreover, for three directions belonging to one of the planes

preserving its analytic character, the angles of the second rotation are found to be identical (i.e., the angles between these directions are preserved).

(K'') The mapping (5.100) preserves the analytic character of three planes ω_1, ω_2, ω_3, and moreover, for three directions belonging to one of the planes preserving its analytic character, the coefficients of the linear distortion are found to be identical.

Hence it follows that if at each point of some domain the mapping (5.100) belongs to the class $\mathcal{C}^1$ and has a nonzero Jacobian and satisfies the condition (K') or (K''), then it is holomorphic in that domain. These requirements may be weakened in view of the theorem on the removal of singularities of analytic functions of several variables (§6.5, Chapter I, (I)).

2. Defect of a nonanalytic plane. Consider the plane

$$w = \alpha z + \beta \bar{z} \tag{5.106}$$

passing through the origin of coordinates. It is nonanalytic for $\beta \neq 0$. Take an arbitrary vector $\{\alpha\gamma + \beta\bar{\gamma}, \gamma\}$ lying on this plane and starting from the origin of coordinates. We construct the analytic plane (defined uniquely) passing through this vector

$$w = \frac{\alpha\gamma + \beta\bar{\gamma}}{\gamma} z \tag{5.106$_1$}$$

and calculate the tangent of the angle between the planes (5.106) and (5.106$_1$). It turns out that this quantity, namely

$$\delta = \frac{2\,|\beta|}{\big||\alpha|^2 - |\beta|^2 + 1\big|}, \tag{5.107}$$

is independent of the choice of the parameter γ (i.e., of the choice of the vector $\{\alpha\gamma + \beta\bar{\gamma}, \gamma\}$ lying in the plane (5.106)). In order that the plane (5.106) should be analytic, it is necessary and sufficient that the quantity (5.107) should be equal to zero. For this reason the expression (5.107) is called the defect of the nonanalytic plane (5.106).

If the mapping (5.100) is monogenic at the point (0, 0), then the mapping (5.101) preserves the analytic character of any plane $w = \omega z$. In the general case this plane goes over into the (generally speaking) nonanalytic plane (5.102). Its defect is

$$\delta(\omega, T) = \frac{2\,|(a\omega + b)(r\bar{\omega} + t) - (c\omega + d)(p\bar{\omega} + q)|}{\big||a\omega + b|^2 + |c\omega + d|^2 - |p\bar{\omega} + q|^2 - |r\bar{\omega} + t|^2\big|}. \tag{5.108}$$

Let equality (5.100) define a mapping $(T) \in \mathcal{C}^1$ in some domain $G \subset C^2$. We shall say that (T) is a *mapping with bounded defect* in the domain G if at all points $(w, z) \in G$

$$\delta(\omega, T) \leq H. \tag{5.109}$$

Here H is some constant quantity. We denote the collection of all such mappings by $D_G(H)$.

3. Quasi-biholomorphic mappings of the first type were first considered by S. Hitotumatu [1]. These are defined by means of quasi-holomorphic functions.

DEFINITION (k-quasi-holomorphic function). A function $f(z)$ is said to be *k-quasi-holomorphic* in a domain $G \subset C_z^n$ if

1) the function $f(z) \in \mathcal{C}^1$ in the domain G;

2) for every point $(z_1^0, \cdots, z_n^0) \in G$ the function

$$F(t) = f(a_1 t + z_1^0, \cdots, a_n t + z_n^0)$$

is k-quasi-conformal in some neighborhood of the point $t = 0$ on the plane C_t^1. Here t is a complex parameter and $a_1, \cdots, a_n$ are complex constants.

REMARK. The function $F(t) \in \mathcal{C}^1$ is said to be k-quasi-holomorphic or k-quasi-conformal in some domain of the plane C_t^1 if in that domain

$$F'_{\bar{t}} = \mu(t) F'_t, \quad \text{where } |\mu(t)| \leq k < 1. \tag{5.110}$$

For $k = 0$ the function $F(t)$ turns out to be holomorphic. It should be noted that the condition (5.110) is often replaced by other equivalent ones.

S. Hitotumatu [1] established a number of properties of k-quasi-holomorphic functions in the space C_z^n. In particular he proved the

THEOREM 26.4. 1) *If a function $f(z) \in \mathcal{C}^1$ satisfies in a domain $G \subset C_z^n$ the system of differential equations*

$$\frac{\partial \bar{f}}{\partial z_j} = \nu(z) \frac{\partial f}{\partial z_j}, \qquad j = 1, \ldots, n, \tag{5.111}$$

where the function $\nu(z)$ is continuous in the domain G and satisfies the condition

$$|\nu(z)| \leq k < 1, \tag{5.112}$$

then the function $f(z)$ is k-quasi-holomorphic in the domain G.

2) *A k-quasi-holomorphic function $f(z)$ in a domain $G \subset C_z^n$ satisfies at every point of the domain G, where* $\operatorname{grad} f \neq 0$, *the system of differential equations* (5.111), *where the function $\nu(z)$ is continuous and satisfies the condition* (5.112)

in the domain G.

DEFINITION (quasi-biholomorphic mapping of the first type). A homeomorphic mapping of a domain $G \subset C^n$ onto a domain $G^* \subset C^n$

$$(T) \qquad z_i^* = f_i(z), \quad i = 1, \cdots, n \tag{5.113}$$

is called a k-quasi-biholomorphic mapping of the first type if all the functions $f_i(z)$ are k-quasi-holomorphic in the domain G.

The above definition implies that the functions $f_i(z)$ defining a quasi-biholomorphic mapping of the first type satisfy the system of differential equations

$$\frac{\partial \overline{f_i}}{\partial z_j} = \nu_i(z) \frac{\partial f_i}{\partial z_j}, \qquad i, j = 1, \ldots, n, \tag{5.111$_1$}$$

where the functions $\nu_i(z)$ satisfy the equations

$$|\nu_i(z)| \leqslant k < 1, \qquad i = 1, \ldots, n \tag{5.112$_1$}$$

(evidently the constant k may be taken identical for all functions $\nu_i(z)$). The collection of all quasi-biholomorphic mappings of the first type of the domain G characterized by the constant k (from the condition (5.112$_1$)) is denoted by $X_G(k)$. Evidently $X_G(0)$ is the collection of all biholomorphic mappings of the domain G.

Starting from the class of mappings $X_G(k)$, one can consider the class of mappings $Y_G(k)$ which is defined in the same way as $X_G(k)$, but with the condition (5.112$_1$) replaced by

$$|\nu_i(z)| \geq k > 1, \quad i = 1, \cdots, n. \tag{5.112$'$}$$

For $k = \infty$ the mapping (5.113) from the class $Y_G(k)$ becomes "antibiholomorphic"; this latter means that in this case the functions $\overline{f_i(z)}$ are holomorphic in the domain G. Therefore the mappings of the class $Y_G(k)$ may be called *quasi-antibiholomorphic mappings of the first type.*

The conditions (5.112$_1$) and (5.112$'$) from the last definitions may be replaced by other equivalent ones. We consider this problem for the space $C^2_{w,z}$ of complex variables w, z. In this case equations (5.113) can be written in the form (5.100) and their differentials at an arbitrary point $(w_0, z_0) \in G$ (which we again take as the origin of coordinates) in the form (5.101). Then the conditions (5.111$_1$) at this point take the form

$$\overline{p} = \nu_1 a, \quad \overline{q} = \nu_1 b, \quad \overline{r} = \nu_2 c, \quad \overline{t} = \nu_2 d, \tag{5.111$_2$}$$

and the conditions (5.112$_1$) and (5.112$'$) become respectively

$$|\nu_i| \leq k < 1, \tag{5.112$_2$}$$

$$|\nu_i| \geq k > 1. \tag{5.112$''$}$$

Here $\nu_i = \nu_i(w_0, z_0)$, $i = 1, 2$. It should also be noted that for $n = 2$ the classes of mappings $X_G(k)$ and $Y_G(k)$ exhaust all mappings (5.113) of the class $\mathcal{C}^1$ that satisfy the condition (5.111) and have a positive orientation (a positive Jacobian).

THEOREM 26.5 (Rudnik [1]). 1) *If a mapping* $T \in X_G(k)$, *then* $T \in D_G(2k/(1-k^2))$;

2) *if a mapping* $T \in Y_G(k)$, *then* $T \in D_G(2k/(k^2-1))$;

3) *if* $T \in D_G(H)$ *and the conditions* (5.111$_1$) *with* $|\nu_i(z)| < 1$, $i = 1, 2$, *are fulfilled for this mapping, then* $T \in X_G(H/\sqrt{1+H^2})$;

4) *if* $T \in D_G(H)$ *and the conditions* (5.111$_1$) *with* $|\nu_i(z)| > 1$, $i = 1, 2$, *are fulfilled for this mapping, then* $T \in Y_G(\sqrt{1+H^2}/H)$.

A number of properties of quasi-biholomorphic mappings of the first type have been established in the papers by Hitotumatu [1], Šmatkov [2] and Rudnik [1].

It may cause some inconvenience to consider quasi-biholomorphic mappings of the first type as the hull of the collection of all biholomorphic mappings. This inconvenience arises because for the superposition of a biholomorphic mapping S: $z^* \longrightarrow z^{**}$ the mapping $S \circ T$, where T is a quasi-biholomorphic mapping defined by equalities (5.113), will in general not be quasi-biholomorphic.

4. Quasi-biholomorphic mappings of the second type have been considered only for domains of the space C^2.

DEFINITION (quasi-biholomorphic mappings of the second type). A homeomorphic mapping of the class $\mathcal{C}^1$ of a domain $G \subset C^2$ onto a domain $G^* \subset C^2$ is called a *quasi-biholomorphic mapping of the second type* if:

1) it preserves at every point of this domain G the analytic character of not less than three planes;

2) it belongs to the collection $D_G(H)$ of all mappings of the domain G with bounded defect.

Here $H > 0$ is a certain number given in advance. The collection of all quasi-biholomorphic mappings of the second type of the domain G that are characterized by the constant H will be denoted by $A_G(H)$.

If the mapping $T \in A_G(H)$ is given by equalities (5.100), while its differential at a point $(w_0, z_0) \in G$ is given by equalities (5.101), then the biholomorphic

mapping

$$(\tau) \qquad \omega = \omega(w, z), \quad \zeta = \zeta(w, z) \tag{5.114}$$

of some neighborhood of the point (w_0, z_0) is said to *correspond to the mapping* (T) at the point (w_0, z_0) if $\omega'_w(w_0, z_0) = a$, $\omega'_z(w_0, z_0) = b$, $\zeta'_w(w_0, z_0) = c$, $\zeta'_z(w_0, z_0) = d$. Of course, such a corresponding biholomorphic mapping is not uniquely defined.

In an analogous way we may define the antibiholomorphic mappings corresponding at the point (w_0, z_0) to the mapping (5.100).

It can be shown that the distortions of lengths, angles, areas, volumes caused by biholomorphic mappings of the second type of the domain G for a sufficiently small H differ little from the distortions of these quantities arising from the corresponding biholomorphic mappings. In addition, the changes of lengths and angles under biholomorphic mappings are naturally characterized by the quantities mentioned in §7.3, Chapter I, (I) (see Theorems 7.5, (I) and 7.6, (I)), while the changes of lengths and angles under quasi-biholomorphic mappings are characterized by the quantities defined analogously.

As an example we shall present the coefficient of distortion of volumes at a point $(w_0, z_0) \in G$ for the mapping $T \in A_G(H)$, defined by equality (5.100), in comparison with the corresponding quantity for the biholomorphic mapping τ defined by equalities (5.114). The first of these is equal to the Jacobian J_0, the second is the quantity j_0^2, where $j_0 = |ad - bc|$. As we have seen above, $J_0 = (1 - \mu\nu)^2 j_0^2$ for the proper choice of coordinate system. Here $\mu > 0$ and $\nu > 0$ are the quantities entering into the expression (5.105). Evidently the coefficient j_0 is identical for all biholomorphic mappings τ corresponding, at the point (w_0, z_0), to the mapping T.

We shall denote by $A'_G(H)$ the collection of all quasi-biholomorphic mappings of the second type that are subject to the supplementary condition $\mu\nu < 1$ (for the geometric meaning of this condition, see the remark after Theorem 26.3; it is easy to see from formula (5.105) that the mappings $A'_G(H) \subset A_G(H)$ and that they also make up a hull of biholomorphic mappings).

We shall denote by $B_G(h)$ the collection of all mappings (5.100) of the domain $G \subset C^2$ that at every point $(w_0, z_0) \in G$: 1) preserve the analytic character of not less than three planes; 2) satisfy the condition

$$1 - \frac{\sqrt{J_0}}{j_0} = \mu\nu \leq h < 1. \tag{5.115}$$

In this case $\mu\nu < 1$; therefore it is meaningful to consider the class of mappings $B_G(h)$ only for the values $h < 1$. We have the

THEOREM 26.6 (Šmatkov [2]) *The following inclusion relation holds*:

$$B_G(h) \subset A'_G\left(\frac{2\sqrt{h}}{1-h}\right); \quad A'_G(H) \subset B_G\left[\frac{H^4}{(1+\sqrt{1+H^2})^2}\right].$$

EXAMPLE. The mapping (see Šmatkov [2])

$$w_1 = \frac{1}{\sqrt[4]{3}}(z_1 + \lambda\bar{z}_1 + 3\lambda\bar{z}_2) + \frac{1}{2}[z_1 + z_2 + 2\lambda(\bar{z}_1 + \bar{z}_2)]^2$$

$$w_2 = \sqrt[4]{3}(z_2 + \lambda\bar{z}_1 - \lambda\bar{z}_2) + \frac{1}{2}[z_1 + z_2 + 2\lambda(\bar{z}_1 + \bar{z}_2)]^2$$

for $|\lambda| < \epsilon/16 < 1/2$ belongs to the class $D(\epsilon)$ in some neighborhood of the origin of coordinates. Its differential for any λ, at all points (z_1^0, z_2^0) of the space $C^2_{z_1, z_2}$, preserves the analytic character of the planes $(z_1 - z_1^0) = \omega(z_2 - z_2^0)$ for which $|\omega - 1| = 4$.

We note that, besides the mappings we have considered here, many authors have investigated other collections of quasi-biholomorphic mappings (see, for example, Bergman [12,13]).

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